Survey of Meshless and Generalized Finite Element Methods: A Unified Approach

Ivo Babuška *   Uday Banerjee †   John E. Osborn ‡

Abstract

In the last few years meshless methods for numerically solving partial differential equations came into the focus of interest, especially in the engineering community. This class of methods was essentially stimulated by difficulties related to mesh generation. Mesh generation is delicate in many situations, *e.g.,* when the domain has complicated geometry; when the mesh changes with time, as in crack propagation, and remeshing is required at each time step; when a Lagrangian formulation is employed, especially with non-linear PDE's. In addition, a need to have flexibility in the selection of approximating functions (*e.g.,* the flexibility to use non-polynomial approximating functions), played a significant role in the development of meshless methods. There are many recent papers, and two books, on meshless methods; most of them are of engineering character, without any mathematical analysis.

In this paper we address meshless methods and the closely related generalized finite element methods for solving linear elliptic equations, using variational principles. We give a unified mathematical theory with proofs, briefly address implementational aspects, present illustrative numerical examples, and provide a list of reference to the current literature.

The aim of the paper is to provide a survey of a part of this new field, with emphasis on mathematics. We present proofs of essential theorems because we feel these proofs are essential for understanding the mathematical aspects of meshless methods, which has approximation theory as a major ingredient. As always, any new field is stimulated by and related to older ideas. This will be visible in our paper.

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*Texas Institute for Computational and Applied Mathematics, ACE 6.412, University of Texas at Austin, Austin, TX 78712. This research was supported by NSF Grant # DMS-98-02367 and ONR Grant # N00014-99-1-0724.

†Department of Mathematics, 215 Carnegie, Syracuse University, Syracuse, NY 13244. E-mail address: banerjee@syr.edu. WWW home page URL: http://bhaskara.syr.edu. This research was partially supported by the TICAM Visiting Faculty Research Fellowship Program.

‡Department of Mathematics, University of Maryland, College Park, MD 20742. E-mail address: jeo@math.umd.edu. WWW home page URL: http://www.math.umd.edu/~jeo. This research was partially supported by the TICAM Visiting Faculty Research Fellowship Program.

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Contents

1 Introduction

a. A Brief Historical Review of the Numerical Solution of Partial Differential Equations.

The numerical solution of partial differential equations has been of central importance for many years. Significant progress has been made in this area, especially in the last 30 years; this progress is directly related to the developments in computer technology. Methods such as, for example, the finite Element Method, are used in all fields of application.

Although significant progress has been made, numerical methods for the solution of differential equations are still often based on heuristic ideas, and verified by numerical experiments. Mathematical analysis is often shallow, and fails to fully address important issues that arise in the application of the methods to important problems in engineering and science.

There are three classical families of numerical methods for solving PDEs:

1. Finite Difference Methods

2. Finite Volume Methods

3. Finite Element Methods

These three families have two common, basic features:

1. They employ a mesh;

2. They use local approximation by polynomials.

We discuss each of these features in turn.

Mesh generation is often very expensive—especially in human cost. This is for several reasons for this cost:

· The domain of the underlying problem can have very complex geometry.

· The domain of the problem may change with time, which requires remeshing at each time step, as for example in the problem of crack propagation or when Lagrangian coordinates are used.

· Adaptive procedures require change of mesh during computation.

Although large progress has been made in the theory and practice of mesh generation, the construction of the mesh is still a very delicate component of the numerical solution of differential equations. For this reason there is an interest in the development of methods that eliminate or reduce the need for a mesh.

Although polynomials have outstanding approximation properties, there are situations in which they are not effective. We mention problems whose solutions are not smooth in the sense that they may not have several bounded derivatives. For such problems, there are sometimes other approximating functions, which we will refer to as *special*, that are effective. The classical methods are not flexible in this regard: they do not use these special non polynomial approximating functions. There is thus an interest in developing and analyzing methods that can flexibly use these special approximation functions.

This created the need to develop methods that address both of these issues—the elimination, completely or partially, of the need for mesh; and the effective use of special (non polynomial) special approximating functions. The inspiration for such methods came mainly from two sources.

The first of these sources is the class of classical particle methods that arise in physical simulation in connection with the Boltzmann Equation or with fluid dynamics. Particle methods attempt to describe the motion of the atoms or their averages (or their density) in Lagrangian Coordinates (see [39], [40], [65], [66], [68], [69], for example).

The other source is the idea of interpolation in the context of the general *Variational Methods* (of Galerkin type). With these methods one has a finite dimensional space, *the trial space*, the method selects an approximation from this space, and, under certain general conditions, it is known that the error in the approximation is no larger than a constant times the error in best approximation by functions in the trial space. Thus the quality of the method is determined by the approximation property of the trial space. It is thus natural to try to find a trial space that has good approximation properties. This property relates directly to interpolation by the approximating functions. For functions in one dimension this was a classical issue in numerical analysis, and starting around 1950 was studied in higher dimension and for an arbitrary distribution of points. It was recognized that the construction of trial spaces could be based on the idea of interpolation.

b. Meshless Methods.

Let us now make the discussion of Variational Methods more precise. We consider an elliptic PDE, which has the variational or weak form,

$$u \in H_1; \qquad B(u,v) = \mathcal{F}(v), \text{ for all } v \in H_2, \tag{1.1}$$

where H_1, H_2 are two Hilbert spaces, $B(u,v)$ is a bounded bilinear form on $H_1 \times H_2$, and $\mathcal{F}(v)$ is a continuous linear functional on H_2. Under certain general condition (the inf-sup or BB condition; see [5],[11]), the solution u is characterized by (1.1). We are interested in approximating u. Toward that end, we assume we have two finite dimensional space $M_1 \subset H_1, M_2 \subset H_2$ that satisfy the discrete inf-sup condition (see [11]). The approximate solution u_{M_1} is characterized by

$$u_{M_1} \in M_1; \qquad B(u_{M_1}, v) = \mathcal{F}(v), \text{ for all } v \in M_2. \tag{1.2}$$

As a consequence of the fact that M_1 and M_2 satisfy the discrete inf-sup condition, we know that the approximation u_{M_1} is quasi-optimal, *i.e.,*

$$\|u - u_{M_1}\|_{H_1} \le C \inf_{\chi \in M_1} \|u - \chi\|_{H_1}. \tag{1.3}$$

We note that there are delicate problems related to the discrete inf-sup condition when the spaces M_1 and M_2 are different; as, *e.g.,* in the case of mixed methods. In [3] different spaces are used (without mathematical analysis of the discrete inf-sup condition).

We thus see that the quality of the approximation, *i.e.,* the error $\|u - u_{M_1}\|_{H_1}$ is mainly determined by the approximation property of the trial space M_1, *i.e.,* by

$$E_1 = \inf_{\chi \in M_1} \|u - \chi\|_{H_1}.$$

It is thus natural to select the trial space M_1 so that E_1 is small. To do this effectively one should use whatever information is available on the solution u. Note that with a general variational method, as we have formulated it, there is no mention of a mesh. Of course, one may use a mesh to construct a good trial space; that, in fact, is exactly what is done with a usual finite element method. For example, the trial space is the space of piecewise linear functions over a mesh.

Meshless Methods, however, either avoid the use of a mesh, or use a mesh only minimally, for example, only for the numerical integration. The Petrov-Galerkin method given by (1.2) is a meshless method if the construction of M_1 and M_2 either does not require a mesh or requires a mesh only minimally. Thus, in designing Meshless Methods within the framework of variational methods, we have two general goals:

1. The construction of trial spaces M_1 that effectively approximate the solution, and the construction of test spaces M_2 ensuring inf-sup (stability) condition.

   > If the solution has special features, *e.g.,* if it is not smooth, we should have the flexibility to use special approximating functions.

2. The minimizing of the need for a mesh.

   > In meshless methods, there is sometime a mesh in the background, used for numerical integration, but one may not need a mesh generator.

We note that there are meshless methods that are not of the type given by (1.2), *e.g.,* methods based on collocation, but the construction of approximating space follows the guidelines of the construction of M_1 as mentioned before.

The approximating (trial) spaces, M_1, can be the spans of specific approximating functions (shape functions), with either global or local supports. Polynomials and radial functions are examples of approximating functions with global supports. See [63] for a discussion of the use of polynomials and [27], [74] for a discussion of the use of radial functions. Another type of approximating functions is related to interpolation and data fitting procedures. For a survey of various approaches we refer to [3], [31], [37], [38], [41], [53], [54], [61], and [77]. Typical finite element approximating functions and spline functions have local supports. In [16] shape functions that are effective for the approximation of solutions of elliptic equations with rough coefficient were identified and analyzed; the idea in [16] was extended and developed in [18]. The approximating functions used in [16] and [17] can be characterized as solutions of particular homogeneous differential equations. In one dimension, L-splines – a generalization of splines, satisfy a differential equation and are used as approximating functions ; see, *e.g.* [87]. Principles for the selection of shape function were addresses in [12].

We note that in the engineering literature many names are used for methods that differ only in the construction of shape functions, or in their implementation; see, *e.g.,* [30] and others ([84]). For a survey of results on Meshless Methods we refer to [15], [22], [34], [42], [56], [57], and [76].

One of the major problems of meshless methods is the imposition of boundary conditions, especially the Dirichlet boundary conditions. It is well known that if the underlying problem is a Dirichlet BVP, the essential boundary condition is addressed with a method such as the penalty method or the method of Lagrange multipliers. On the other hand, the boundary condition of a Neumann Problem is natural and doesn't need to be explicitly imposed in the variational formulation. In both the situations, a simple uniform mesh on a rectangle containing the domain can be used; the mesh need not conform to the boundary and a mesh generator is not needed. These ideas are classical and have been extensively analyzed (for example, see [11]). These ideas of imposing boundary conditions can be used in the context of meshless methods and this approach was also mentioned in [56]. The boundary of the domain does come into play in the construction of the stiffness matrix, but a mesh generator is not needed. This approach was generalized and used together with the ideas in [16], [17], and [83] in solving problems with very complex geometries; see, *e.g.,* [82].

We mention finally a meshless method—The Generalized Finite Element Method (GFEM)—that attempts to simultaneously achieve the two goals of variationally formulated meshless methods. With this method one begins with a *Partition of Unity.* Construction of a partition of unity is a relatively simple task. It can be done by various means. One is to use a simple mesh, for example, a uniform mesh, and use the associated hat functions as the partition of unity. We could also use ideas from various interpolation procedures, *e.g.* the Shepard Method. It is essential that the construction can, but is not required to, utilize the geometry of the domain. The partition of unity on the domain is obtained by restriction. The partition of unity functions typically have compact supports with small diameters.

Then we multiply the partition of unity functions by functions that are defined separately and independently on the supports of the partition functions. In this way we create shape functions that belong to $H^1(\Omega)$, and can be used in the variational method. We thereby obtain a large flexibility in the construction of shape functions, and the associated trial spaces. This flexibility can be used to construct approximations that utilize the available information, the character of a singularity, or a boundary layer, *e.g.*, on the approximated function (solution). Hence the method achieves the goals mentioned above.

We do face three serious difficulties in the implementation of the GFEM. First there is the problem of numerical integrations when the areas over which we integrate are not simple triangles, simplicies, *etc.*, as with the usual FEM. We note, however, that the process is completely parallelizable. A second difficulty is the treatment of essential boundary conditions. The third issue concerns the system of linear equations. It may be singular, and thus certain classical methods, such as multigrid, may not be applicable. These difficulties can be, and have been, overcome in some implementations, so it is clear that the GFEM shows a definite advantage over the classical FEM in certain situations. We mention problems with complex geometry, crack propagation, and analysis of mulit-site local damage.

Of course, any new method should be compared with previously developed methods, and the class of problems for which the new method is superior should be identified. Theoretical and practical experience (see [15], [56], [83]) is progressing in this direction. Meshless Methods in various forms, *e.g.,* within the framework of collocation or variational methods, are now the subject of many papers and (engineering) books, which mainly focus on practical aspects without serious theoretical analysis.

This paper focuses on ideas and theoretical results. Some are adjustments of old ideas and results. Some results are based on papers that are submitted or in the final stage of preparation. Although we focus on the theory, we have attempted to address theoretical issues that illuminate practical issues. We will show that the results presented here are natural generalizations of the classical FEM, which is a special case of some of the methods presented here. This paper addresses only problems related to linear PDEs.

Various relevant and typical references are provided. The reference list is not comprehensive, although, together with the citations in the references provide, in our opinion, a very reasonable description of the current state of the art for meshless methods.

c. The Scope of this Paper.

The short Section 2 defines the model problem, a linear elliptic boundary value problem. Section 3.1 presents approximation results when the particles are uniformly distributed. The presented results were obtained using the Fourier Transform. Section 3.2 presents an alternate proof of the approximation results that can be generalized to the case of non-uniformly distributed particles. Section 3.3 discusses approximation for arbitrarily distributed particles. Section 4 discusses the construction of shape functions, and presents some results on interpolation and on the asymptotic form of the error. Section 5 addresses the question of superconvergence. Section 6 discusses the Generalized Finite Element Method. Section 7 discusses the application of the approximation results developed in Sections 3, and discusses the treatment of Dirichlet boundary conditions. Section 8 explains some implementational aspects. Section 9 reports some numerical examples obtained by the GFEM, when the domain is very complex. Finally, Section 10 presents additional results and challenges.

2 The Model Problem

For concreteness and simplicity we will address the weak solution of the model problem

$$-\Delta u + u = f(x), \text{ on } \Omega \subset R^n \tag{2.1}$$

and

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \tag{2.2}$$

or

$$u = 0 \text{ on } \partial\Omega, \tag{2.3}$$

where $f \in L_2(\Omega)$ is given. We will assume that Ω is a Lipschitz domain; additional assumptions on $\partial\Omega$ will given as needed.

The weak solution $u_0 \in H^1(\Omega)$ ($H_0^1(\Omega)$, respectively) satisfies

$$B(u_0, v) = \mathcal{F}(v), \text{ for all } v \in H^1(\Omega)\, (v \in H_0^1(\Omega), \text{respectively}), \tag{2.4}$$

where

$$B(u, v) \equiv \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx \text{ and } \mathcal{F}(v) \equiv \int_\Omega f v\, dx. \tag{2.5}$$

The energy norm of u_0 is defined by

$$\|u_0\|_E \equiv B(u_0, u_0)^{1/2} = \|u_0\|_{H^1(\Omega)}. \tag{2.6}$$

We will write H instead of $H^1(\Omega)$ or $H_0^1(\Omega)$ if no misunderstanding can occur.

Let $S \subset H$ be a finite dimensional subspace, called the approximation space. Then the Galerkin approximation, $u_S \in S$, to u_0 is determined by

$$\tilde{B}(u_S, v) = \mathcal{F}(v), \text{ for all } v \in S, \tag{2.7}$$

where $\tilde{B}$ is either B or a perturbation of B. If $\tilde{B} = B$, it is immediate that

$$\|u_0 - u_S\|_{H^1(\Omega)} = \inf_{\chi \in S} \|u_0 - \chi\|_{H^1(\Omega)}. \tag{2.8}$$

Hence, the main problem is the approximation of u_0 by functions in S.

Remark 2.1 The Finite Element Method (FEM) is the Galerkin Method where S is the span of functions with small supports. For the history of the FEM, see [9] and the reference therein.

Remark 2.2 The classical Ritz method uses spaces of polynomials on Ω for the approximation spaces; see, *e.g.,* [63].

As mentioned above, the Finite Element Method uses basis functions with small supports, *e.g.*, "hill" functions. The theory of approximation with general hill functions with translation invariant supports was developed in 1970 in [4] using the Fourier Transform. The results in [4] were applied to the numercial solution of PDE in [5]. A very similar theory, also based on the Fourier Transform, was later developed in [80] and [81]; see also [55]. Later, hill functions were, in another context, called *Particle Functions* (see [39]). In the 1990s, hill functions began to be used in the framework of *meshless methods.* For a broad survey of meshless methods see [56]. A survey of the approximation properties of radial hill functions is given in [27].

In this paper we will survey basic meshless approximation results and their use in the framework of Galerkin Methods.

3 Approximation by Local Functions in R^n; the h-version Analysis

As mentioned in Section 2, we are interested in the approximation of functions by particle shape functions. We first consider uniformly distributed particles, and then general—non-uniformly distributed—particles.

3.1 Uniformly Distributed Particles and Associated Particle Shape Functions

Let

$$\mathbb{Z}^n \equiv \{j = (j_1, \ldots, j_n) : j_1, \ldots, j_n \text{ integers}\}$$

be the integer lattice, and, for $j = (j_1, \cdots, j_n) \in \mathbb{Z}^n$ and $0 < h \leq 1$, let

$$x_j^h = (j_1 h, \ldots, j_n h) = hj.$$

The points x_j^h are called *uniformly distributed particles.* When considering such families of particles, we often construct associated shape functions as follows. Let $\phi \in H^q(\mathbb{R}^n)$, for some $0 \leq q$, be a function with compact support; let $\eta \equiv$ supp ϕ, and suppose

$$\eta \subset B_\rho = \{x \in \mathbb{R}^n : \|x\|^2 = x_1^2 + \cdots + x_n^2 < \rho\}.$$

We assume that $0 \in \mathring{\eta}$ ($\mathring{\eta}$ is the interior of η). Then define

$$\phi_j^h(x) = \phi_j^h(x_1, \cdots, x_n) = \phi\left(\frac{x - jh}{h}\right) = \phi\left(\frac{x_1 - j_1 h}{h}, \ldots, \frac{x_n - j_n h}{h}\right), \quad (3.1)$$

for $j \in \mathbb{Z}^n$ and $0 < h \leq 1$. Clearly,

$$\eta_j^h \equiv \text{ supp } \phi_j^h = \left\{x : \frac{x - jh}{h} \in \eta\right\} \subset B_{\rho h}^j = \{x : \|x - x_j^h\| < \rho h\},$$

and $x_j^h \in \mathring{\eta}_j^h$. Particle and particle shape functions defined in this way are *translation invariant* in the sense that

$$x_{j+l}^h = x_j^h + x_l^h \text{ and } \phi_{j+l}^h(x) = \phi_j^h(x - x_l^h),$$

and will sometimes be referred to as translation invariant. They are a special case of general (non-uniformly distributed) particles, which will be addressed in Section 3.3. We refer to h as the size of the particle and the function ϕ is called the *basic shape function.* In this section we will be interested in the approximation properties of

$$V_h^{k,q} \equiv \left\{v = v(x) = \sum_{j \in \mathbb{Z}^n} w_j^h \phi_j^h(x) : w_j^h \in \mathbb{R}\right\}, \quad (3.2)$$

which is the linear span of the associated shape functions, as $h \to 0$. The parameter k in $V_h^{k,q}$ is related to a property of $\phi_j^h(x)$'s, which will be discussed later. We will refer to $V_h^{k,q}$ as the particle space in $\mathbb{R}^n$. The $w_j^h{}'s$ are called *weights.* Specifically, given $u \in H^{k+1}(\mathbb{R}^n)$, we are interested in estimating

$$\inf_{\chi \in V_h^{k,q}} \|u - \chi\|_{H^s(\mathbb{R}^n)}, \tag{3.3}$$

for $0 \le s \le \min\{q, k+1\}$. We are especially interested in the maximum μ such that

$$\inf_{\chi \in V_h^{k,q}} \|u - \chi\|_{H^s(\mathbb{R}^n)} \le C(k,q) h^\mu \|u\|_{H^{k+1}(\mathbb{R}^n)}, \tag{3.4}$$

for $0 \le s \le \min\{q, k+1\}$, where the constant $C = C(k,q)$ depends on k, q, but is independent of h (C also depends on ϕ).

Because we are assuming the particles are uniformly distributed, and hence the particles and shape functions are translation invariant, estimates of the form (3.4) can be obtained via the Fourier Transform. This was done in [4] and [80], [81]. We will cite one of the results in [81].

Let

$$\hat{\phi}(\xi) = \hat{\phi}(\xi_1, \cdots, \xi_n) \equiv \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot\xi}\, dx$$

denote the Fourier Transform of $\phi(x)$. We note that $\hat{\phi}(\xi) \in C^\infty(\mathbb{R}^n)$ since $\phi(x)$ has compact support. We use the usual multi-index notation: $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \ge 0$; $|\alpha| = \alpha_1 + \cdots + \alpha_n$; $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; and

$$D^\alpha \hat{\phi} = \frac{\partial^{|\alpha|} \hat{\phi}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}}.$$

Theorem 3.1 *[81] Suppose $\phi \in H^q(\mathbb{R}^n)$ has compact support, where the smoothness index $q \ge 0$ is an integer. Then the following three conditions are are equivalent:*

1. $$\hat{\phi}(0) \neq 0 \tag{3.5}$$

   and

   $$D^\alpha \hat{\phi}(2\pi j) = 0, \text{ for } 0 \neq j \in \mathbb{Z}^n \text{ and } |\alpha| \le k, \tag{3.6}$$

   where k is a non-negative integer.

2. *For $|\alpha| \le k$,*

   $$\sum_{j \in \mathbb{Z}^n} j^\alpha \phi(x - j) = dx^\alpha + q^\alpha(x), \text{ for all } x \in \mathbb{R}^n, \tag{3.7}$$

   where $d \neq 0$ and $q^\alpha(x)$ is a polynomial with degree $< |\alpha|$.

   The equality in (3.7) is equality in $L_2(\mathbb{R}^n)$, i.e., equality for almost all $x \in \mathbb{R}^n$. The function of the right-hand side of (3.7) is, of course, continuous. If the function on the left-hand side is continuous, which will be the case if $q > n/2$, then (3.7) will hold for all $x \in \mathbb{R}^n$.

3. *For each $u \in H^{k+1}(\mathbb{R}^n)$ there are weights $w_j^h \in \mathbb{R}$, for $j \in \mathbb{Z}^n$ and $0 < h$, such that*

$$\begin{aligned}\|u \quad - & \sum_{j\in\mathbb{Z}^n} w_j^h \phi_j^h\|_{H^s(\mathbb{R}^n)} \\ \leq & \quad Ch^{k+1-s}\|u\|_{H^{k+1}(\mathbb{R}^n)}, \ \text{for } 0 \leq s \leq \min\{q, k+1\}, \end{aligned} \tag{3.8}$$

and

$$h^n \sum_{j\in\mathbb{Z}^n} (w_j^h)^2 \leq K^2 \|u\|^2_{H^0(\mathbb{R}^n)}. \tag{3.9}$$

Here C and K may depend on q, k, and s, but are independent of u and h. The exponent $k+1-s$ is the best possible if k is the largest integer for which (3.7) holds.

If (3.7) holds, the basic shape function ϕ is called *Quasi-Reproducing of Order k*. If (3.7) holds with $d = 1$ and $q^\alpha(x) = 0$, ϕ is called *Reproducing of Order k*. If ϕ is quasi-reproducing of order k (respectively, reproducing of order k), then the corresponding particle shape functions ϕ_i^h are also called quasi-reproducing of order k (respectively, reproducing of order k). The parameter k in $V_h^{k,q}$, defined in (3.2), is the quasi-reproducing order of the basic shape function ϕ.

Remark 3.1 If one were to define the notion of quasi-reproducing basic shape function ϕ of order k with the formula

$$\sum_{j\in\mathbb{Z}^n} j^\alpha \phi(x-j) = d_\alpha x^\alpha + q^\alpha(x), \ \text{for all } x \in \mathbb{R}^n, \ \text{for } |\alpha| \leq k, \tag{3.10}$$

where $d_\alpha \neq 0$, it might appear that this would lead to a larger class of ϕ's. This, however, is not the case; it is easily shown that if ϕ satisfies (3.10), then $d_\alpha = d$, for $|\alpha| \leq k$.

In one dimension we can prove more.

Theorem 3.2 *[81] Suppose $\phi \in H^q(\mathbb{R})$ has compact support and satisfies Condition 1 in Theorem 3.1, i.e., satisfies (3.5) and (3.6). Then*

$$\hat{\phi}(\xi) = Z(\xi)\left(\frac{\sin(\xi/2)}{\xi/2}\right)^{k+1}, \tag{3.11}$$

where $Z(\xi)$ is an entire function.

Proof. Because ϕ has compact support, $\hat{\phi}(\xi)$ is an entire function, and because of (3.5) and (3.6), $\hat{\phi}(0) \neq 0$ and $\hat{\phi}(\xi)$ has zeros of at least order k at $2\pi j$, $0 \neq j \in \mathbb{Z}$. Let

$$\hat{\sigma}_k(\xi) = \left(\frac{\sin(\xi/2)}{\xi/2}\right)^{k+1}. \tag{3.12}$$

The function $\hat{\sigma}_k(\xi)$ is entire with only zeros of order $k+1$ at $2\pi j$, for $0 \neq j \in \mathcal{Z}$. Hence

$$Z(\xi) = \hat{\phi}(\xi)/\hat{\sigma}_k(\xi)$$

is entire, and

$$\hat{\phi}(\xi) = Z(\xi) \left(\frac{\sin(\xi/2)}{\xi/2} \right)^{k+1}, \tag{3.13}$$

as desired. □

Theorem 3.3 *[4] Suppose $\phi \in H^q(\mathbb{R})$ has compact support and satisfies Condition 1 in Theorem 3.1, i.e., satisfies (3.5) and (3.6). Then, for any $\epsilon > 0$,*

$$\textit{supp}\ \phi \not\subset \left[-\frac{(k+1)}{2} + \epsilon, \frac{(k+1)}{2} - \epsilon \right]. \tag{3.14}$$

Proof. Suppose, on the contrary, that

$$\text{supp}\ \phi \subset [-(k+1)/2 + \epsilon, (k+1)/2 - \epsilon], \text{ for some } \epsilon > 0. \tag{3.15}$$

We will show that this assumption leads to a contradiction.

The function $\hat{\phi}(\xi)$ is entire and, with $\xi = \xi_1 + i\xi_2$, (3.15) implies

$$|\hat{\phi}(\xi)| \leq C e^{(\frac{(k+1)}{2} - \epsilon)|\xi_2|}. \tag{3.16}$$

This estimate follows directly from the definition of the Fourier Transform and assumption (3.15). Using elementary properties of the sine function, we find that

$$\left| \left(\frac{\sin(\xi/2)}{\xi/2} \right)^{k+1} \right| \geq C \frac{e^{\frac{k+1}{2}|\xi_2|}}{|\xi_2|^{k+1}}, \text{ for } |\xi_2| \text{ large}. \tag{3.17}$$

Using (3.5), (3.6), (3.16), and (3.17), we have

$$|Z(\xi)| = \left| \frac{\hat{\phi}(\xi)}{\left(\frac{\sin(\xi/2}{\xi/2} \right)^{k+1}} \right| \leq C_0 + C_{k+1}|\xi|^{k+1}, \text{ for all } \xi \in \mathcal{C}, \tag{3.18}$$

where $Z(\xi)$ is as in (3.11). Since $Z(\xi)$ is entire, estimate (3.18) implies (via a generalization of Liouville's Theorem for entire functions) that $Z(\xi)$ is a polynomial of degree $\leq k+1$. Next, we use (3.11) and (3.16) to get

$$\left| Z(\xi) \left(\frac{\sin(\xi/2)}{\xi/2} \right)^{k+1} \right| = |\hat{\phi}(\xi)| \leq C e^{(\frac{k+1}{2} - \epsilon)|\xi_2|}. \tag{3.19}$$

Combining this estimate with the lower bound in (3.17) we have

$$|Z(\xi)| \leq C|\xi|^{k+1} e^{-\epsilon|\xi_2|}, \text{ for } |\xi_2| \text{ large}. \tag{3.20}$$

This implies $Z(\xi) = 0$. Thus, (3.11) implies $\hat{\phi}(\xi) = 0$, which contradicts (3.5). Thus (3.15) is false, which proves (3.14). □

The case of uniformly distribute particles is very special, but we have considered it, and cited Theorem 3.1 from [81] because the result provides necessary and sufficient conditions on the basic shape function ϕ for the validity of the approximability result (3.8) and (3.9), leading to the optimal value for μ in (3.4).

<u>Comments on Theorems 3.1–3.3</u>

1. $\hat{\phi}(0) \neq 0$ means that $\int_{\mathbb{R}^n} \phi(x)\, dx \neq 0$.

2. For the validity of (3.8) and (3.9) we need a polynomial reproducing property, namely that given in Condition 2. Note that this conditions differs from the usual polynomial reproducing property, which reads

$$\sum_{j \in \mathbb{Z}^n} j^\alpha \phi(x - j) = x^\alpha, \text{ for all } |\alpha| \leq p,$$

   or, equivalently,

$$\sum_{j \in \mathbb{Z}^n} p(j)\phi(x - j) = p(x), \text{ for all polynomials } p(x) \text{ of degree } \leq k.$$

3. Condition 2 implies that

$$\sum_{j \in \mathbb{Z}^n} \phi(x - j) = d. \tag{3.21}$$

   Hence the functions $\frac{\phi(x-j)}{d}$, $j \in \mathbb{Z}^n$, are a *partition of unity*. The sets $\mathring{\eta}_j^h$ are an open cover of $\mathbb{R}^n$.

4. Taking $q = k + 1$ allows application of Theorem 3.1 for all $s \leq k + 1$. Taking $q > k + 1$, *i.e.*, assuming extra smoothness on the particle shape functions does not change the estimate.

5. Condition (3.9) is a *stability condition*.

6. The simplest basic shape function $\phi(x)$ corresponds to

$$\hat{\phi}(\xi) = \prod_{i=1}^{n} \left(\frac{\sin \xi_i/2}{\xi_i/2} \right)^{k+1}. \tag{3.22}$$

   Thus they (the ϕ's) are spline functions over a rectangular mesh, which are convolutions of the characteristic function of the set $Q^{(n)} = (-1/2, 1/2)^n$.

7. In one dimension ($n = 1$) the simplest basic shape function corresponds to
$$\hat{\sigma}_k(\xi) = \left(\frac{\sin \xi/2}{\xi/2}\right)^{k+1}, \tag{3.23}$$
and they are B-splines. Furthermore, as shown in Theorem 3.2, if $\phi(x)$ satisfies (3.5) and (3.6), then $\hat{\phi}(\xi)$ is the product of $\hat{\sigma}_k(\xi)$ and a suitable entire function $Z(\xi)$. Taking account of the growth of $Z(\xi)$, we see that it is the Fourier transform of a function with compact support. Writing $Z(\xi) = \hat{\psi}(\xi)$, we can express (3.11) as
$$\hat{\phi}(\xi) = \hat{\psi}(\xi)\hat{\sigma}_k(\xi).$$
Thus any $\phi(x)$ that satisfies (3.5) and (3.6), which may or may not be piecewise polynomial, can be constructed via convolution of B-splines with functions of compact support. If $n > 1$, no such divisor $\hat{\phi}/\hat{\sigma}_k$ exits in general.

8. Theorem 3.3 has as especially simple interpretation for ϕ's that satisfies (3.5) and (3.6), and whose support is an interval. Namely, supp $\phi \supset \left[-\frac{k+1}{2}, \frac{k+1}{2}\right]$, and hence grows with k. As is well known, the support of the B-spline of order k is $\left[-\frac{k+1}{2}, \frac{k+1}{2}\right]$, and hence it has a minimal support.

9. The space $V_h^{k,q}$ is a S^{t,k^*}*−regular system* (this notion will be introduced in Section 3.2), with $k^* = q$ and $t = k + 1$. S^{t,k^*}-regular systems are analyzed in [11]. They have many important properties, some of which will be used in the following sections.

10. The approximability of the classical finite element shape functions (the hat functions) can be analyzed with Theorem 3.1 with $q = k = 1$.

11. The weights in (3.8) depend on u, but they are not unique. We note that the functions ϕ_j^h may be linearly dependent.

3.2 Alternate Proof for Uniformly Distributed Particles and Particle Shape Functions

In this section we first give an alternative proof that Condition 2 in Theorem 3.1 implies estimate (3.8), again for uniformly distributed particles and associated shape functions. This alternative proof does not use the Fourier Transform, and it can be naturally generalized to the non-uniformly distributed particles situation.

We review our notation before stating the theorem. Recall that
$$x_j^h = jh, \text{ for } j \in \mathbb{Z}^n \text{ and } 0 < h,$$

are the particles, and $\phi \in H^q(\mathbb{R}^n)$, with $q \geq 0$, is the basic shape function. Also $\eta =$ supp $\phi \subset B_\rho$, and $0 \in \mathring{\eta}$. Then the particle shape functions, $\phi_j^h(x)$, are defined by

$$\phi_j^h(x) = \phi\left(\frac{x - jh}{h}\right);$$

it is immediate that

$$\eta_j^h = \text{ supp } \phi_j^h \subset B_{\rho h}^j,$$

and $x_j^h \in \mathring{\eta}_j^h$.

Theorem 3.4 *Suppose $\phi \in H^q(\mathbb{R}^n)$, with smoothness index $q \geq 0$, has compact support $\eta \subset B_\rho$, and suppose the $\phi_j^h(x)$ are defined in (3.1). Suppose $k = 0, 1, 2, \cdots$ and suppose, for $|\alpha| \leq k$,*

$$\sum_{j\in\mathbb{Z}^n} j^\alpha \phi(x - j) = dx^\alpha + q^\alpha(x); \tag{3.24}$$

here $d \neq 0$ and $q^\alpha(x)$ is a polynomial of degree $< |\alpha|$, i.e., suppose ϕ is quasi-reproducing of order k. Suppose u satisfies

$$\sum_{j\in\mathbb{Z}^n} \|u\|^2_{H^{r_j+1}(B^j_{\overline{\rho}h})} < \infty, \text{ where } 0 \leq r_j \leq k, \tag{3.25}$$

where $\overline{\rho} \geq 1$ is sufficiently large and independent of h. Then there exist weights w_l^h such that

$$\|u - \sum_{l\in\mathbb{Z}^n} w_l^h \phi_l^h\|^2_{H^s(\mathbb{R}^n)} \leq C \sum_{j\in\mathbb{Z}^n} h^{2(r_j+1-s)} \|u\|^2_{H^{r_j+1}(B^j_{\overline{\rho}h})}, \text{ for } 0 \leq s \leq \min_{j\in\mathbb{Z}_n}\{q, r_j+1\}, \tag{3.26}$$

where C is independent of u and h. If $u \in H^{k'+1}(\mathbb{R}^n)$, where $0 \leq k' \leq k$, then

$$\|u - \sum_{l\in\mathbb{Z}^n} w_l^h \phi_l^h\|_{H^s(\mathbb{R}^n)} \leq Ch^{k'+1-s}\|u\|_{H^{k'+1}(\mathbb{R}^n)}, \text{ for } 0 \leq s \leq \min\{q, k'+1\}. \tag{3.27}$$

Proof. The proof is in several steps.

1. Suppose ϕ satisfies (3.24), and write $q^\alpha(x) = \sum_{|\gamma|\leq|\alpha|-1} d_{\gamma\alpha} x^\gamma$. Then

$$\begin{aligned}
\sum_{j\in\mathbb{Z}^n} (x_j^h)^\alpha \phi_j^h(x) &= \sum_{j\in\mathbb{Z}^n} (jh)^\alpha \phi_j^h(x) \\
&= h^{|\alpha|} \sum_{j\in\mathbb{Z}^n} j^\alpha \phi\left(\frac{x}{h} - j\right) \\
&= h^{|\alpha|}\left\{d\left(\frac{x}{h}\right)^\alpha + q^\alpha\left(\frac{x}{h}\right)\right\} \\
&= d\,x^\alpha + h^{|\alpha|} \sum_{|\gamma|\leq|\alpha|-1} d_{\gamma\alpha}\left(\frac{x}{h}\right)^\gamma \\
&= d\,x^\alpha + \sum_{|\gamma|\leq|\alpha|-1} h^{|\alpha|-|\gamma|} d_{\gamma\alpha} x^\gamma, \quad \text{for } |\alpha| \leq k.
\end{aligned} \tag{3.28}$$

Equations (3.24) and (3.28) are, in fact, equivalent; (3.28) could be viewed as a scaled version of (3.24). For any $p \in \mathcal{P}^k$, there is a uniquely determined $w = w_{p,h} \in \mathcal{P}^k$ satisfying

$$p(x) = \sum_{j\in\mathbb{Z}^n} w_{p,h}(x_j^h)\phi_j^h(x), \quad \text{for all } x \in \mathbb{R}^n. \tag{3.29}$$

We first prove the existence of $w_{p,h}$, and begin by considering the monic polynomials: $p_\alpha = x^\alpha$. Suppose $|\alpha| = 0$. Then from (3.28) we have

$$1 = \sum_{j\in\mathbb{Z}^n} \frac{1}{d}\,\phi_j^h(x) = \sum_{j\in\mathbb{Z}^n} w_{\{1\},h}(x_j^h)\,\phi_j^h(x),$$

where $w_{\{1\},h}(x) = 1/d$. Next suppose $|\alpha| = 1$. Using (3.28) again we have

$$x^\alpha = \sum_{j\in\mathbb{Z}^n} \frac{1}{d}(x_j^h)^\alpha\phi_j^h(x) - \frac{d_{0\alpha}}{d} = \sum_{j\in\mathbb{Z}^n} w_{\{x^\alpha\},h}(x_j^h)\phi_j^h(x)$$

where

$$w_{\{x^\alpha\},h}(x) = \frac{x^\alpha}{d} - \frac{hd_{0\alpha}}{d^2}.$$

Proceeding in this way, by induction, we get $w_{\{x^\alpha\},h}(x)$ for $|\alpha| \le k$, where $w_{\{x^\alpha\},h}(x)$ is of the form

$$w_{\{x^\alpha\},h}(x) = e_{\alpha\alpha}x^\alpha + \sum_{|\beta|\le|\alpha|-1} e_{\alpha\beta}h^{|\alpha|-|\beta|}x^\beta, \tag{3.30}$$

where $e_{\alpha\alpha} = d^{-1}$ and $e_{\alpha\beta}$ are expressions in $d_{\gamma\alpha}$, $|\gamma| < |\alpha|$. For $p(x) = \sum_{|\alpha|\le k} c_\alpha x^\alpha$, we let $w_{p,h}(x) = \sum_{|\alpha|\le k} c_\alpha w_{\{x^\alpha\},h}(x)$. It is immediate that

$$p(x) = \sum_{j\in\mathbb{Z}^n} w_{p,h}(x_j^h)\phi_j^h(x),$$

which establishes the existence of $w_{p,h}(x)$. One can show that

$$w_{p,h}(x) = \sum_{|\beta|\le k}\Big[\sum_{|\beta|+1\le|\alpha|\le k,\ \alpha=\beta} c_\alpha d_{\alpha\beta}h^{|\alpha|-|\beta|}\Big]x^\beta. \tag{3.31}$$

To prove the uniqueness, suppose $w_{p,h}(x) = 0$. We will show that $p(x) = \sum_{|\alpha|\le k} c_\alpha x^\alpha = 0$. Since $w_{p,h}(x) = 0$, it is clear from (3.31) that the coefficient of x^β is zero for $|\beta| \le k$, from which we can deduce that $c_\alpha = 0$, $|\alpha| \le k$, and thus $p(x) = 0$. It will be convenient to write $w_{p,h}(x) = \mathcal{A}^h p$. Then $\mathcal{A}^h : \mathcal{P}^k \to \mathcal{P}^k$ is a 1–1, onto mapping satisfying

$$p(x) = \sum_{j\in\mathbb{Z}^n} (\mathcal{A}^h p)(x_j^h)\phi_j^h(x), \quad \text{for all } x \in \mathbb{R}^n, \text{ for any } p \in \mathcal{P}^k. \tag{3.32}$$

We define $\mathcal{A} = \mathcal{A}^h$ when $h = 1$. We note that $\mathcal{A}$ satisfies (3.32) with $h = 1$. We also have

$$[(\mathcal{A}^h)^{-1}w](x) = \sum_{j\in\mathbb{Z}^n} w(x_j^h)\phi_j^h, \qquad \text{for all } x \in \mathbb{R}^n, \text{ for any } w \in \mathcal{P}^k. \tag{3.33}$$

It is also clear from the construction of that $\mathcal{A}^h : \mathcal{P}^i \to \mathcal{P}^i$, for $i \leq k$.

2. Define the cells ω_j and ω_j^h:

$$\omega_j = \{x : \|x - j\|_\infty \equiv \max_{i=1,\ldots,n} |x_i - j_i| < \rho\}$$

and

$$\omega_j^h = \{x : \|x - x_j^h\|_\infty \equiv \max_{i=1,\ldots,n} |x_i - x_{j_i}^h| < \rho h\}.$$

The families $\{\omega_j\}_{j\in\mathbb{Z}^n}$ and $\{\omega_j^h\}_{j\in\mathbb{Z}^n}$ are open covers of $\mathbb{R}^n$ provided $\rho > 1/2$. Let

$$A_j^h = \{l \in \mathbb{Z}^n : \eta_l^h \cap \omega_j^h \neq \emptyset\},$$

and define

$$\Omega_j^h = \cup_{l\in A_j^h} \omega_l^h.$$

It is immediate that one can select M and $\bar{\rho}$ such that

$$\text{card } A_j^h \leq M \tag{3.34}$$

and

$$\Omega_j^h \subset B_{\bar{\rho}h}^j. \tag{3.35}$$

The constants M and $\bar{\rho}$ are independent of j and h, but do depend on ϕ; specifically on ρ.

For any $l \in \mathbb{Z}^n$, since $u \in H^{r_l+1}(B_{\bar{\rho}h}^l)$, it is well known ([25],[26],[29]) that there is a polynomial $p^{l,h} = p_k^{l,h}$ of degree $\leq k$ such that

$$\|u - p^{l,h}\|_{H^s(B_{\bar{\rho}h}^l)} \leq Ch^{r_l+1-s}\|u\|_{H^{r_l+1}(B_{\bar{\rho}h}^l)}, \text{ for } 0 \leq s \leq r_l + 1 \leq k + 1, \tag{3.36}$$

where C is independent of u, h, and l, but does depend on k ($p^{l,h}$ can, in fact, be chosen such that its degree $\leq r_l$). Define the weights

$$w_l^h = (\mathcal{A}^h p^{l,h})(x_l^h). \tag{3.37}$$

Let j be fixed. We will work with the polynomial $p^{j,h}$, which satisfies (3.36) with $l = j$, as well as the polynomial $p^{l,h}$. Using (3.37), we find

$$\begin{aligned}
&\|u - \sum_{l\in\mathbb{Z}^n} w_l^h\phi_l^h\|_{H^s(\omega_j^h)} \\
&\quad \leq \|u - \sum_{l\in A_j^h} w_l^h\phi_l^h\|_{H^s(\omega_j^h)} \\
&\quad \leq \|u - \sum_{l\in A_j^h} (\mathcal{A}^h p^{j,h})(x_l^h)\phi_l^h\|_{H^s(\omega_j^h)} \\
&\qquad + \sum_{l\in A_j^h} |(\mathcal{A}^h p^{j,h})(x_l^h) - (\mathcal{A}^h p^{l,h})(x_l^h)|\, \|\phi_l^h\|_{H^s(\omega_j^h)}.
\end{aligned} \tag{3.38}$$

We now estimate the two terms on the right side of (3.38).

3. From (3.32) and the definition of A^h_j, we have

$$p(x) = \sum_{l\in\mathbb{Z}^n} (\mathcal{A}^h p)(x^h_l)\phi^h_l(x) = \sum_{l\in A^h_j} (\mathcal{A}^h p)(x^h_l)\phi^h_l(x), \text{ for } x \in \omega^h_j,$$

for any $p \in \mathcal{P}^k$. Using this formula and (3.36) with $l = j$, we obtain the estimate

$$\begin{aligned} \|u - \sum_{l\in A^h_j} (\mathcal{A}^h p^{j,h})(x^h_l)\phi^h_l\|_{H^s(\omega^h_j)} &= \|u - p^{j,h}\|_{H^s(\omega^h_j)} \\ &\le Ch^{r_j+1-s}\|u\|_{H^{r_j+1}(B^j_{\bar\rho h})}, \end{aligned} \tag{3.39}$$

for the first term of (3.38).

A scaling argument shows that

$$\|\phi^h_l\|_{H^s(\omega^h_j)} \le h^{-s+n/2}\|\phi\|_{H^s(\mathbb{R}^n)}.$$

Thus

$$\begin{aligned} &\sum_{l\in A^h_j} |\mathcal{A}^h p^{j,h}(x^h_l) - \mathcal{A}^h p^{l,h}(x^h_l)|\, \|\phi^h_l\|_{H^s(\omega^h_j)} \\ &\qquad \le Ch^{-s+n/2} \sum_{l\in A^h_j} |\mathcal{A}^h p^{j,h}(x^h_l) - \mathcal{A}^h p^{l,h}(x^h_l)|. \end{aligned} \tag{3.40}$$

It remains to estimate the right side of this inequality.

For $l \in A^h_j$, $\omega^h_l \subset \Omega^h_j$, and hence, using (3.35), $\omega^h_l \subset B^j_{\bar\rho h}$. Also $\omega^h_l \subset B^l_{\bar\rho h}$. Thus, using (3.36) with $s = 0$, we have

$$\begin{aligned} \|p^{j,h} - p^{l,h}\|_{H^0(\omega^h_l)} &\le \|p^{j,h} - u\|_{H^0(\omega^h_l)} + \|u - p^{l,h}\|_{H^0(\omega^h_l)} \\ &\le \|p^{j,h} - u\|_{H^0(B^j_{\bar\rho h})} + \|u - p^{l,h}\|_{H^0(B^l_{\bar\rho h})} \\ &\le Ch^{r_j+1}\|u\|_{H^{r_j+1}(B^j_{\bar\rho h})} \\ &\qquad + Ch^{r_l+1}\|u\|_{H^{r_l+1}(B^l_{\bar\rho h})}. \end{aligned} \tag{3.41}$$

It is easily shown that there is a constant C such that

$$\|w\|_{L^\infty(\omega^h_l)} \le Ch^{-n/2}\|w\|_{H^0(\omega^h_l)}, \text{ for any } w \in \mathcal{P}^k; \tag{3.42}$$

C is independent of w, h, and l. Applying (3.42) to $w = \mathcal{A}^h p^{j,h} - \mathcal{A}^h p^{l,h}$, we have

$$\begin{aligned} &|(\mathcal{A}^h p^{j,h})(x^h_l) - (\mathcal{A}^h p^{l,h})(x^h_l)| \\ &\quad \le \|\mathcal{A}^h p^{j,h} - \mathcal{A}^h p^{l,h}\|_{L^\infty(\omega^h_l)} \\ &\quad \le Ch^{-n/2}\|\mathcal{A}^h p^{j,h} - \mathcal{A}^h p^{l,h}\|_{H^0(\omega^h_l)}. \end{aligned} \tag{3.43}$$

For any $p \in \mathcal{P}^k$, we write $p(x) = \tilde{p}\left(\frac{x - x_l^h}{h}\right)$ where $\tilde{p} \in \mathcal{P}^k$. Using (3.32) with $h = 1$ (recall that $\mathcal{A} = \mathcal{A}^h$ for $h = 1$), we see that

$$\tilde{p}(x) = \sum_{i \in \mathbb{Z}^n} (\mathcal{A}\tilde{p})(i)\phi(x - i),$$

and therefore

$$\begin{aligned}
p(x) &= \tilde{p}\left(\frac{x - x_l^h}{h}\right) \\
&= \sum_{i \in \mathbb{Z}^n} (\mathcal{A}\tilde{p})(i)\phi\left(\frac{x - x_{i+l}^h}{h}\right) \\
&= \sum_{i \in \mathbb{Z}^n} (\mathcal{A}\tilde{p})(i)\phi_{i+l}^h(x) \\
&= \sum_{i \in \mathbb{Z}^n} (\mathcal{A}\tilde{p})(i - l)\phi_i^h(x) \\
&= \sum_{i \in \mathbb{Z}^n} (\mathcal{A}\tilde{p})\left(\frac{x_i^h - x_l^h}{h}\right)\phi_i^h(x)
\end{aligned}$$

Comparing the above expression with (3.32) and using the uniqueness of the representation (3.29), we obtain

$$(\mathcal{A}^h p)(x) = (\mathcal{A}\tilde{p})\left(\frac{x - x_l^h}{h}\right). \tag{3.44}$$

We further note that $\|\mathcal{A}\tilde{p}\|_{H^0(\omega_0)}$ is a norm on $\tilde{p}$, and since all norms are equivalent on a finite dimensional space, we have

$$\|\mathcal{A}\tilde{p}\|_{H^0(\omega_0)} \leq C\|\tilde{p}\|_{H^0(\omega_0)}. \tag{3.45}$$

Therefore from (3.44) and (3.45), and using the transformation $y = (x - x_l^h)/h$, we have

$$\begin{aligned}
\|\mathcal{A}^h p\|_{H^0(\omega_l^h)}^2 = \int_{\omega_l^h} |(\mathcal{A}^h p(x)|^2\, dx &= \int_{\omega_0} |(\mathcal{A}\tilde{p}(y)|^2\, dy \\
&\leq C\int_{\omega_0} |\tilde{p}(y)|^2\, dy \\
&= C\int_{\omega_l^h} |\tilde{p}((x - x_l^h)/h)|^2\, dx \\
&= C\int_{\omega_l^h} |p(x)|^2\, dx \\
&= C\|p\|_{H^0(\omega_l^h)}^2, \quad \text{for } p \in \mathcal{P}^k,
\end{aligned} \tag{3.46}$$

with C independent of p, l, and h. Combining (3.43), (3.46) with $p^{l,h} - p^{j,h}$, and (3.41) yields

$$
\begin{aligned}
&|(\mathcal{A}^h p^{j,h})(x_l^h) - (\mathcal{A}^h p^{l,h})(x_l^h)| \\
&\qquad \le Ch^{-n/2}(h^{r_j+1}\|u\|_{H^{r_j+1}(B^j_{\bar\rho h})} + h^{r_l+1}\|u\|_{H^{r_l+1}(B^l_{\bar\rho h})}),
\end{aligned}
$$

and hence, using (3.34), we have the estimate

$$
\begin{aligned}
&\sum_{l\in A_j^h} |(\mathcal{A}^h p^{j,h})(x_l^h) - (\mathcal{A}^h p^{l,h})(x_l^h)| \\
&\qquad \le Ch^{-n/2}\{Mh^{r_j+1}\|u\|_{H^{r_j+1}(B^j_{\bar\rho h})} \\
&\qquad\qquad + \sum_{l\in A_j^h} h^{r_l+1}\|u\|_{H^{r_l+1}(B^l_{\bar\rho h})}\} \qquad (3.47)
\end{aligned}
$$

for the right side of (3.38). Now we combine (3.38), (3.39), (3.40), and (3.47) to obtain

$$
\|u - \sum_{l\in\mathbb{Z}^n} w_l^h \phi_l^h\|_{H^s(\omega_j^h)} \le C \sum_{l\in A_j^h} h^{r_l+1-s}\|u\|_{H^{r_l+1}(B^l_{\bar\rho h})}. \qquad (3.48)
$$

4. Finally, we estimate $\|u - \sum_{l\in\mathbb{Z}^n} w_l^h \phi_l^h\|_{H^s(\mathbb{R}^n)}$. Using (3.48), which is valid for all $j\in\mathbb{Z}^n$, and (3.34) we obtain

$$
\begin{aligned}
\|u - \sum_{l\in\mathbb{Z}^n} w_l^h \phi_l^h\|^2_{H^s(\mathbb{R}^n)} &\le \sum_{j\in\mathbb{Z}^n} \|u - \sum_{l\in\mathbb{Z}^n} w_l^h \phi_l^h\|^2_{H^s(\omega_j^h)} \\
&\le CM \sum_{j\in\mathbb{Z}^n} \sum_{l\in\mathcal{A}_j^h} h^{2(r_l+1-s)}\|u\|^2_{H^{r_l+1}(B^l_{\bar\rho h})} \\
&\le C \sum_{j\in\mathbb{Z}^n} h^{2(r_j+1-s)}\|u\|^2_{H^{r_j+1}(B^j_{\bar\rho h})}, \qquad (3.49)
\end{aligned}
$$

where C is independent of u and h. This proves (3.26).

Suppose $u \in H^{k'+1}(\mathbb{R}^n)$, where $0 \le k' \le k$. Then taking $r_j = k'$ in (3.49), and using the fact that the overlap in $\{B^j_{\bar\rho h}\}_{j\in\mathbb{Z}^n}$ is bounded independently of h, we get

$$
\begin{aligned}
\|u - \sum_{l\in\mathbb{Z}^n} w_l^h \phi_l^h\|_{H^s(\mathbb{R}^n)} &\le Ch^{k'+1-s}\left(\sum_{j\in\mathbb{Z}^n} \|u\|^2_{H^{k'+1}(B^j_{\bar\rho h})}\right)^{1/2} \qquad (3.50) \\
&\le Ch^{k'+1-s}\|u\|_{H^{k'+1}(\mathbb{R}^n)},
\end{aligned}
$$

where C is independent of u and h, which is (3.27). □

Remark 3.2 Estimates (3.26) and (3.27) have been established provided $\bar\rho$ is sufficiently large; specifically, provided (3.34) holds. As pointed out in connection with (3.35), $\bar\rho$ depends on ρ. Note that the constants C in (3.26) and (3.27) depend on $\bar\rho$.

So far in this section, we have considered functions u defined on $\mathbb{R}^n$, and have presented a result on the approximation of u by particle shape functions. We now consider functions u defined on a bounded domain Ω in $\mathbb{R}^n$ with Lipschitz continuous boundary. We will show that $V_h^{k,q}$, defined in (3.2), when restricted to Ω, provides accurate approximation to u.

We first recall the well known extension result ([78]), that there is a bounded extension operator $E : L_2(\Omega) \to L_2(\mathbb{R}^n)$, *i.e.*, an operator E satisfying $Eu|_\Omega = u$ for all $u \in L_2(\Omega)$, such that if $u \in H^m(\Omega)$, then $Eu \in H^m(\mathbb{R}^n)$ and

$$\|Eu\|_{H^m(\mathbb{R}^n)} \le C_m \|u\|_{H^m(\Omega)}, \qquad \text{for all } u \in H^m(\Omega),\ m = 0,\ 1,\ \ldots. \tag{3.51}$$

Here C_m is independent of u but depends on m.

We define a subset of $\mathbb{Z}^n_\Omega$ of $\mathbb{Z}^n$, which will be used in the next result, by

$$\mathbb{Z}^n_\Omega = \{j \in \mathbb{Z}^n \,:\, \mathring{\eta}_j^h \cap \Omega \ne \emptyset\}, \tag{3.52}$$

where $\eta_j^h = \text{supp } \phi_j^h$.

Theorem 3.5 *Suppose $\phi \in H^q(\mathbb{R}^n)$, with smoothness index $q \ge 0$, has compact support $\eta \subset B_\rho$, and is quasi-reproducing of order k. Suppose $u \in H^{k'+1}(\Omega)$, where $0 \le k' \le k$. Then there are weights w_j^h such that*

$$\|u - \sum_{l \in \mathbb{Z}^n_\Omega} w_l^h \phi_l^h\|_{H^s(\Omega)} \le C h^{k'+1-s} \|u\|_{H^{k'+1}(\Omega)}, \quad 0 \le s \le \min(q, k'+1), \tag{3.53}$$

where C is independent of u and h.

Proof. Suppose $u \in H^{k'+1}(\Omega)$, and let $\bar{u} = Eu$, where E is the extension operator mentioned above. Applying (3.27) of Theorem 3.4 to $\bar{u}$, there are weights w_l^h such that

$$\|\bar{u} - \sum_{l \in \mathbb{Z}^n} w_l^h \phi_l^h\|_{H^s(\mathbb{R}^n)} \le C h^{k'+1-s} \|\bar{u}\|_{H^{k'+1}(\mathbb{R}^n)}.$$

Therefore, from (3.51) with $m = k' + 1$, we have

$$\begin{aligned} \|u - \sum_{l \in \mathbb{Z}^n} w_l^h \phi_l^h\|_{H^s(\Omega)} &= \|\bar{u} - \sum_{l \in \mathbb{Z}^n} w_l^h \phi_l^h\|_{H^s(\Omega)} \\ &\le \|\bar{u} - \sum_{l \in \mathbb{Z}^n} w_l^h \phi_l^h\|_{H^s(\mathbb{R}^n)} \\ &\le C h^{k'+1-s} \|\bar{u}\|_{H^{k'+1}(\mathbb{R}^n)} \\ &\le C h^{k'+1-s} \|u\|_{H^{k'+1}(\Omega)}. \end{aligned} \tag{3.54}$$

From the definition of $\mathbb{Z}^h_\Omega$ in (3.52), it is clear that

$$\|u - \sum_{l \in \mathbb{Z}^n} w_l^h \phi_l^h\|_{H^s(\Omega)} = \|u - \sum_{l \in \mathbb{Z}^n_\Omega} w_l^h \phi_l^h\|_{H^s(\Omega)},$$

and therefore from (3.54), we get the desired result. □

By examining the approximation of u, obtained in Theorem 3.5, namely

$$\sum_{l\in\mathbb{Z}^n_\Omega} w_l^h \phi_l^h\big|_\Omega,$$

we see that the sum involves only those l's for which

$$\text{supp } \phi_l^h \cap \Omega \neq \emptyset,$$

i.e., only those particles x_l^h such that $\text{dist}(x_l^h,\Omega) < \rho h$. So the approximation involves particle shape functions corresponding to the particles inside Ω, as well as some particles lying outside Ω. We will denote the span of these shape functions by

$$V^{k,q}_{\Omega,h} = \text{ span}\{\phi_j^h\big|_\Omega : \text{supp } \phi_j^h \cap \Omega \neq \emptyset\}. \tag{3.55}$$

Thus the functions in $V^{k,q}_{\Omega,h}$ are the functions in $V^{k,q}_h$ restricted to Ω.

$(\mathbf{t},\mathbf{k}^*)$-regular systems:

We now introduce the notion of a (t,k^*)–regular system of functions (*cf.* [11]). Let $\Omega \subseteq \mathbb{R}^n$, and suppose $S_h(\Omega)$, $0 < h \leq 1$, is a one-parameter family of linear spaces of functions on Ω. For $0 \leq k^* \leq t$, $S_h(\Omega)$ will be called a (t,k^*)*–regular system* and will be denoted by $S_h^{t,k^*}(\Omega)$ if

1. $S_h^{t,k^*}(\Omega) \subset H^{k^*}(\Omega), \quad \text{for } 0 < h \leq 1.$ (3.56)
2. For every $u \in H^l(\Omega)$, with $0 \leq l$, there is a function $g_h \in S_h^{t,k^*}$ such that
$$\|u - g_h\|_{H^s(\Omega)} \leq Ch^\mu \|u\|_{H^l(\Omega)}, \text{ for } 0 \leq s \leq \min\{l,k^*\}, \tag{3.57}$$
where $\mu = \min\{t-s, l-s\}$. The constant C is independent of u and h.

We now introduce two additional notions.

(LA) A (t,k^*)–regular system $S_h^{t,k^*}(\Omega)$ is said to satisfy a *local assumption* if for $u \in H^l(\Omega)$, with support S, the function $g_h \in S_h^{t,k^*}(\Omega)$ in (3.57) can be chosen so that the support S_h of g_h has the property that

$$S_h \subset S^{\lambda h} \equiv \{x \in \Omega : d(x,S) \leq \lambda h\},$$

where $d(x,S)$ is the distance from x to S, and λ is independent of h.

(IA) We say that $S_h^{t,k^*}(\Omega)$ satisfies an *inverse assumption* (*cf.* [11]) if there is an $0 \leq \epsilon \leq k^*$ such that

$$\|g\|_{H^{k^*}(\Omega)} \leq Ch^{-(k^*-r)}\|g\|_{H^r(\Omega)}, \text{ for all } k^*-\epsilon \leq r \leq k^* \text{ and all } g \in S_h^{t,k^*}(\Omega),$$

where C is independent of h and g (it may depend on k^* and ϵ).

A (t, k^*)-regular system is known as (t, k)-regular system in classical literature ([11]). We have decided to use k^* instead of k in this paper for notational clarity.

The approximation space $V^{k,q}_{\Omega,h}$, defined in (3.55), is a (t, k^*)–regular system; more precisely we have,

Theorem 3.6 *Suppose $0 \leq q < k+1$, and suppose $\phi \in H^q(\mathbb{R}^n)$ has compact support and is quasi-reproducing of order k. Then $V^{k,q}_{\Omega,h}$ is a $(k+1, q)$–regular system.*

Proof. We show that $V^{k,q}_h$ is a (t, k^*)–regular system with $t = k+1$ and $k^* = q$. Since $\phi \in H^q(\mathbb{R}^n)$, it is clear that $V^{k,q}_{\Omega,h} \subset H^q(\Omega)$ and thus (3.56) is immediate with $k^* = q$. Next we show that (3.57) is satisfied. Suppose $u \in H^l(\mathbb{R}^n)$ with $l \geq 0$. If $l = 0$, (3.57) is trivial. So, suppose $1 \leq l$. Applying Theorem 3.5, specifically (3.53) with $k' = \min(k+1, l) - 1$, we get

$$\|u - \sum_{j \in \mathbb{Z}^n_\Omega} w^h_j \phi^h_j\|_{H^s(\Omega)} \leq Ch^{\min(l-s, k+1-s)} \|u\|_{H^{\min(l,k+1)}(\Omega)} \leq Ch^\mu \|u\|_{H^l(\mathbb{R}^n)},$$

for $0 \leq s \leq \min\{q, \min\{l, k+1\}\} = \min\{l, q\}$ (since $q < k+1$), where $\mu = \min\{k+1-s, l-s\}$. This is (3.57), with $g_h = \sum_{l \in \mathbb{Z}^n_\Omega} w^h_l \phi^h_l\big|_\Omega$, $t = k+1$, and $k^* = q$. □

We now show that $V^{k,q}_{\Omega,h}$ satisfies the local assumption (LA).

Theorem 3.7 *Suppose $\phi \in H^q(\mathbb{R}^n)$, where $0 \leq q \leq k+1$, has compact support and is quasi-reproducing of order k. Then $V^{k,q}_{\Omega,h}$ satisfies the local assumption (LA).*

Proof. Suppose $u \in H^l(\Omega)$ such that $\operatorname{supp} u = S \subset \Omega$. Consider the approximation of u, obtained in Theorem 3.5, namely

$$g_h = \sum_{j \in \mathbb{Z}^n_\Omega} w^h_j \phi^h_j. \tag{3.58}$$

A careful study of the proofs of Theorems 3.5 and 3.4, and considering the zero extension of u outside Ω, reveals that for $j \in \mathbb{Z}^n_\Omega$,

$$w^h_j = 0, \quad \text{if and only if } B^j_{\bar{\rho}h} \cap S = \emptyset.$$

Now for $j \in \mathbb{Z}^n_\Omega$ such that $w^h_j \neq 0$, we know that $\eta^h_j = \operatorname{supp} \phi^h_j \subset B^j_{\rho h}$. Therefore, $S_h = \operatorname{supp} g_h = \{x \in \mathbb{R}^n : d(x, S) \leq (\bar{\rho} + \rho)h\}$, and so we can take $\lambda = (\bar{\rho} + \rho)$ in the definition of LA. For small h, we have $S_h \subset \Omega$. Hence $V^{k,q}_{\Omega,h}$ satisfies the local assumption LA. □

Remark 3.3 The particle space $V^{k,q}_h$ is $(k+1, q)$-regular and satisfies the local assumption, LA, for $\Omega = \mathbb{R}^n$.

We note that the particle spaces $V^{k,q}_{\Omega,h}$ and $V^{k,q}_h$ will also satisfy the inverse assumption IA, if additional conditions are imposed on the shape functions $\{\phi^h_j\}$. We will formulate these conditions in Section 3.3 in the context of non-uniformly distributed particles; the corresponding conditions on the shape functions associated with uniformly distributed particles can then be obtained as a special case.

3.3 Approximation by Particle Shape Functions Associated with Arbitrary (Non-Uniformly Distributed) Particles in $\mathbb{R}^n$. The h-version

In this section we will generalize the major part of Theorem 3.4.

Suppose $\{X^\nu\}_{\nu\in N}$ is a family of countable subsets of $\mathbb{R}^n$; the family is indexed by the parameter ν, which varies over the index set N. The points in X^ν are called *particles*, and will be denoted by $\underline{x}$, to distinguish them from general points in $\mathbb{R}^n$. If it is necessary to underline that $\underline{x} \in X^\nu$, we will write $\underline{x} = \underline{x}^\nu$. To each $\underline{x}^\nu \in X^\nu$ we associate

- $h^\nu_{\underline{x}^\nu} = h^\nu_{\underline{x}}$, a positive number;
- $\omega^\nu_{\underline{x}^\nu} = \omega^\nu_{\underline{x}}$, a bounded domain in $\mathbb{R}^n$;
- $\phi^\nu_{\underline{x}^\nu} = \phi^\nu_{\underline{x}}$, a function in $H^q(\mathbb{R}^n)$, with $q \ge 0$ and with $\eta^\nu_{\underline{x}^\nu} = \eta^\nu_{\underline{x}} \equiv \text{supp}\ \phi^\nu_{\underline{x}^\nu}$ assumed compact.

The numbers $h^\nu_{\underline{x}^\nu} = h^\nu_{\underline{x}}$ will be referred to as the *sizes* of the particles $\underline{x}$, and the functions $\phi^\nu_{\underline{x}^\nu}$ are called the *particle shape functions*. For a given $\nu \in N$, let

$$\mathcal{M}^\nu = \left\{ X^\nu, \{h^\nu_{\underline{x}}, \omega^\nu_{\underline{x}}, \phi^\nu_{\underline{x}}\}_{\underline{x}\in X^\nu} \right\}.$$

$\mathcal{M}^\nu$ will be referred to as *particle-shape function system* — and $\{\mathcal{M}^\nu\}_{\nu\in N}$ as a family of particle-shape function systems. This nomenclature is similar to that used in FEM when we speak of a triangulation and a family of triangulation.

Regarding the particle-shape function system, we make several assumptions:

A1. For each ν,

$$\bigcup_{\underline{x}\in X^\nu} \omega^\nu_{\underline{x}} = \mathbb{R}^n,$$

i.e., for each ν, $\{\omega^\nu_{\underline{x}}\}_{\underline{x}\in X^\nu}$ is an open cover of $\mathbb{R}^n$.

A2. For $\underline{x} \in X^\nu$, let

$$S^\nu_{\underline{x}} \equiv \{\underline{y} \in X^\nu : \omega^\nu_{\underline{x}} \cap \omega^\nu_{\underline{y}} \ne \emptyset\}.$$

There is a constant $\kappa < \infty$, which may depend on $\{\mathcal{M}^\nu\}_{\nu\in N}$, but neither on ν, nor on $\underline{x} \in X^\nu$, such that

$$\text{card}\ S^\nu_{\underline{x}} \le \kappa, \text{ for all } \underline{x} \in X^\nu \text{ and all } \nu \in N.$$

A3. For all $\underline{x} \in X^\nu$, for all $\nu \in N$, $\underline{x} \in \mathring{\eta}^\nu_{\underline{x}}$ and $\mathring{\eta}^\nu_{\underline{x}} \subset \omega^\nu_{\underline{x}}$.

A4. For $\underline{x} \in X^\nu$, let

$$\Omega^\nu_{\underline{x}} = \bigcup_{\underline{y} \in Q^\nu_{\underline{x}}} \omega^\nu_{\underline{y}},$$

where

$$Q^\nu_{\underline{x}} \equiv \{\underline{y} \in X^\nu : \eta^\nu_{\underline{y}} \cap \omega^\nu_{\underline{x}} \neq \emptyset\}.$$

There is a $0 < \bar{\rho} < \infty$, which may depend on $\{\mathcal{M}^\nu\}_{\nu \in N}$, but is independent of $\underline{x}$ and ν, such that

$$\Omega^\nu_{\underline{x}} \subset B^{\underline{x}}_{\bar{\rho} h^\nu_{\underline{x}}}, \text{ for all } \underline{x} \in X^\nu, \text{ for all } \nu \in N,$$

where $B^{\underline{x}}_{\bar{\rho} h^\nu_{\underline{x}}}$ is the ball of radius $\bar{\rho} h^\nu_{\underline{x}}$ centered at $\underline{x}$, namely,

$$B^{\underline{x}}_{\bar{\rho} h^\nu_{\underline{x}}} = \{x \in \mathbb{R}^n : \|x - \underline{x}\| \leq \bar{\rho} h^\nu_{\underline{x}}\}.$$

A5. For each $\underline{x} \in X^\nu$, there is a one-to-one mapping $\mathcal{A}^\nu_{\underline{x}} : \mathcal{P}^k \to \mathcal{P}^k$ such that

$$\sum_{\underline{y} \in Q^\nu_{\underline{x}}} (\mathcal{A}^\nu_{\underline{x}} p)(\underline{y}) \phi^\nu_{\underline{y}}(x) = p(x), \text{ for } x \in \omega^\nu_{\underline{x}}, \text{ and any } p \in \mathcal{P}^k, \tag{3.59}$$

and

$$\|\mathcal{A}_{\underline{x}} p\|_{L^2(\omega^\nu_{\underline{y}})} \leq C \|p\|_{L^2(\omega^\nu_{\underline{y}})}, \text{ for all } p \in \mathcal{P}^k, \text{ for all } y \in Q^\nu_{\underline{x}}, \text{ for all } \underline{x} \in X^\nu.$$

A6. For any $0 \leq s \leq q$,

$$\|\phi^\nu_{\underline{y}}\|_{H^s(\omega^\nu_{\underline{z}})} \leq C (h^\nu_{\underline{y}})^{-s+n/2}, \text{ for all } \underline{y} \in Q^\nu_{\underline{x}}.$$

The constant C may depend on $\{\mathcal{M}^\nu\}_{\nu \in N}$, but is independent of $\underline{y}, \underline{x}$, and ν.

A7. There is a constant C such that

$$\|w\|_{L^\infty(\omega^\nu_{\underline{y}})} \leq C (h^\nu_{\underline{y}})^{-n/2} \|w\|_{L^2(\omega^\nu_{\underline{y}})}, \text{ for any } w \in \mathcal{P}^k,$$

where C is independent of $\underline{y}$ and ν.

Remark 3.4 From the definitions of $Q^\nu_{\underline{x}}$ and $S^\nu_{\underline{x}}$, and assumption A3, it is clear that $Q^\nu_{\underline{x}} \subset S^\nu_{\underline{x}}$. Hence from the assumption A2, it is immediate that

$$\text{card } Q^\nu_{\underline{x}} \leq \kappa. \tag{3.60}$$

We could, of course, have stated (3.60) as an assumption, but have chosen to state card $S^\nu_{\underline{x}} \leq \kappa$ as an assumption because, generally, $S^\nu_{\underline{x}}$ is easier to work with than $Q^\nu_{\underline{x}}$. We also note that assumptions A1–A4 imply that, for any $x \in \mathbb{R}^n$,

$$\text{card } \{\underline{x} \in X^\nu : x \in \mathring{\eta}^\nu_{\underline{x}}\} \leq \kappa, \quad \text{for all } \nu \in N. \tag{3.61}$$

Remark 3.5 We note that the left-hand side of (3.59) in A5 is defined for all $x \in \mathbb{R}^n$, but the equality holds only for $x \in \omega^\nu_{\underline{x}}$.

Remark 3.6 Note that assumptions A1–A7 can be thought of as assumptions on $\mathcal{M}^\nu$, for each $\nu \in N$; they are assumptions on $\{\mathcal{M}^\nu\}_{\nu\in N}$ in that they are assumptions on $\mathcal{M}^\nu$ for each ν <u>and</u> that the constants in the assumptions do not depend on ν.

Remark 3.7 Assumptions A5 effectively defines the notion of quasi-reproducing shape functions $\phi^\nu_{\underline{x}}$ of order k. Note that the condition is local: the operator $\mathcal{A}^\nu_{\underline{x}}$ depends on $\underline{x}$, the sum is taken only over $\underline{y} \in Q^\nu_{\underline{x}}$, and the equation holds only for $x \in \omega^\nu_{\underline{x}}$. The shape functions $\phi^\nu_{\underline{x}}$ are said to be reproducing of order k if

$$\sum_{\underline{y}\in X^\nu} p(\underline{y})\phi^\nu_{\underline{y}}(x) = p(x), \text{ for } x \in \mathbb{R}^n, \text{ and any } p \in \mathcal{P}^k.$$

If the shape functions $\phi^\nu_{\underline{x}}$ are reproducing of order k, then it is immediate that they satisfy A5 with $\mathcal{A}^\nu_{\underline{x}}$ equal to the identity mapping for each $\underline{x}$.

Remark 3.8 Assumption A5 implies

$$\bigcup_{\underline{x}\in X^\nu} \mathring{\eta}^\nu_{\underline{x}} = \mathbb{R}^n, \quad \text{for each } \nu.$$

Remark 3.9 Consider uniformly distributed particles, x^h_j, and associated particle shape functions, ϕ^h_j, as defined in Section 3.2. Then with $\underline{x}^\nu = x^h_j$, $h^\nu_{\underline{x}} = h$, $\phi^\nu_{\underline{x}} = \phi^h_j$, and $\omega^\nu_{\underline{x}} = \omega^h_j$, as defined in the proof of Theorem 3.4, the associated particle shape function system satisfy assumptions A1–A7. Note that $\mathcal{A}^\nu_{\underline{x}} = \mathcal{A}^h_{x^h_j} = \mathcal{A}^h$ satisfies A5.

Suppose $\{\mathcal{M}^\nu\}_{\nu\in N}$ is a family of particle shape function systems, satisfying A1–A7. Define

$$\mathbb{V}^{k,q}_\nu = \text{span}\,\{\phi^\nu_{\underline{x}} : \underline{x} \in X^\nu\}, \quad \text{for each } \nu \in N. \tag{3.62}$$

The next theorem gives an approximation error estimate when a function u, defined on $\mathbb{R}^n$, is approximated by a function in $\mathbb{V}^{k,q}_\nu$, $\nu \in N$.

Theorem 3.8 *Suppose the family of particle-shape function systems $\{\mathcal{M}^\nu\}_{\nu\in N}$ satisfies A1–A7, and $h^\nu_{\underline{x}} \le 1$ for $\underline{x} \in X^\nu$, $\nu \in N$. Suppose*

$$\sum_{\underline{x}\in X^\nu} \|u\|^2_{H^{r^\nu_{\underline{x}}+1}(B^{\underline{x}}_{\overline{\rho}h^\nu_{\underline{x}}})} < \infty, \text{ where } r^\nu_{\underline{x}} \le k, \text{ for all } \underline{x} \in X^\nu, \text{ for all } \nu \in N, \tag{3.63}$$

where $\overline{\rho}$ is introduced in A4. Also, suppose that operators $\mathcal{A}^\nu_{\underline{x}}$, introduced in A5, satisfy

$$\|(\mathcal{A}^\nu_{\underline{x}} - \mathcal{A}^\nu_{\underline{y}})p\|_{H^0(\omega^\nu_{\underline{y}})} \le C(h^\nu_{\underline{x}})^{r^\nu_{\underline{x}}+1}\|p\|_{H^0(\omega^\nu_{\underline{y}})}, \text{ for all } p \in \mathcal{P}^k, \ \underline{y} \in Q^\nu_{\underline{x}}, \tag{3.64}$$

for all $\underline{x} \in X^\nu$, $\nu \in N$, where C is independent of $\underline{x}$ and ν. Then there are weights $w^\nu_{\underline{y}} \in \mathbb{R}$, for $\underline{y} \in X^\nu$, for all $\nu \in N$, such that

$$\begin{aligned} \|u \quad - &\sum_{\underline{y}\in X^\nu} w^\nu_{\underline{y}} \phi^\nu_{\underline{y}}\|_{H^s(\mathbb{R}^n)} \\ \leq \quad & C\Big(\sum_{\underline{y}\in X^\nu} (h^\nu_{\underline{y}})^{2(r^\nu_{\underline{y}}+1-s)} \|u\|^2_{H^{r^\nu_{\underline{y}}+1}(B^{\underline{y}}_{\bar\rho h^\nu_{\underline{y}}})}\Big)^{1/2}, \end{aligned} \tag{3.65}$$

for $0 \leq s \leq \inf\{q, r^\nu_{\underline{y}} + 1 : \underline{y} \in X^\nu, \nu \in N\}$. The constant C depends on the constants in Assumptions A1–A7 and on (3.64), but neither on u, nor on ν.

Note: If $\phi^\nu_{\underline{x}}$'s are reproducing of order k, then (3.64) is trivially satisfied (*cf.* Remark 3.7). Since in practice, mainly shape functions, that are reproducing of order k, are used, we have not included (3.64) in the set of basic assumptions (A1–A7).

Proof . The proof of this result is parallel to the proof of Theorem 3.4.

1. The sets $\omega^\nu_{\underline{x}}$ play the role of the sets ω^h_j in the proof of Theorem 3.4. The sets $Q^\nu_{\underline{x}}$, $\Omega^\nu_{\underline{x}}$, $B^{\underline{x}}_{\bar\rho h^\nu_{\underline{x}}}$, and the mapping $\mathcal{A}^\nu_{\underline{x}}$ play the roles of the sets $\mathcal{A}^h_j, \Omega^h_j, B^j_{\bar\rho h}$, and the mapping $\mathcal{A}^h_j$, respectively, in the proof on Theorem 3.4. Assumptions A1–A7 state the properties of these sets and the mappings we will need in this proof.

2. For any $\underline{y} \in X^\nu$, since $u \in H^{r^\nu_{\underline{y}}+1}(B^{\underline{y}}_{\bar\rho h^\nu_{\underline{y}}})$, it is well-known that there is a polynomial $p^{\underline{y},\nu} = p^{\underline{y},\nu}_k$ of degree $\leq k$, such that

$$\|u - p^{\underline{y},\nu}_k\|_{H^s(B^{\underline{y}}_{\bar\rho h^\nu_{\underline{y}}})} \leq C(h^\nu_{\underline{y}})^{r^\nu_{\underline{y}}+1-s}\|u\|_{H^{r^\nu_{\underline{y}}+1}(B^{\underline{y}}_{\bar\rho h^\nu_{\underline{y}}})}, \tag{3.66}$$

for $0 \leq s \leq r^\nu_{\underline{y}} + 1 \leq k+1$, where C is independent of u, ν and $\underline{y}$, but does depend on k ($p^{\underline{y},\nu}_k$ can, in fact, be selected so that its degree $\leq r^\nu_{\underline{y}}$). Define the weights

$$w^\nu_{\underline{y}} = (\mathcal{A}^\nu_{\underline{y}} p^{\underline{y},\nu})(\underline{y}), \tag{3.67}$$

where $\mathcal{A}^\nu_{\underline{y}}$ is the operator introduced in assumption A5.

Let $\underline{x} \in X^\nu$ be fixed. We will work with the polynomial $p^{\underline{x},\nu}$, which satisfies (3.66) with $\underline{y} = \underline{x}$, as well as the polynomial $p^{\underline{y},\nu}$. Using (3.67) we find

$$\begin{aligned} &\|u - \sum_{\underline{y}\in X^\nu} w^\nu_{\underline{y}}\phi^\nu_{\underline{y}}\|_{H^s(\omega^\nu_{\underline{x}})} \\ &\leq \|u - \sum_{\underline{y}\in Q^\nu_{\underline{x}}} w^\nu_{\underline{y}}\phi^\nu_{\underline{y}}\|_{H^s(\omega^\nu_{\underline{x}})} \\ &\leq \|u - \sum_{\underline{y}\in Q^\nu_{\underline{x}}} (\mathcal{A}^\nu_{\underline{x}} p^{\underline{x},\nu})(\underline{y})\, \phi^\nu_{\underline{y}}\|_{H^s(\omega^\nu_{\underline{x}})} \\ &\quad + \sum_{\underline{y}\in Q^\nu_{\underline{x}}} \big|(\mathcal{A}^\nu_{\underline{x}} p^{\underline{x},\nu})(\underline{y}) - (\mathcal{A}^\nu_{\underline{y}} p^{\underline{y},\nu})(\underline{y})\big|\, \|\phi^\nu_{\underline{y}}\|_{H^s(\omega^\nu_{\underline{x}})}. \end{aligned} \tag{3.68}$$

We now estimate the two terms on the right-hand side of (3.68).

3. From the assumption A5, we know that

$$\sum_{\underline{y}\in Q^\nu_{\underline{x}}} (\mathcal{A}^\nu_{\underline{x}} p)(\underline{y})\phi^\nu_{\underline{y}}(x) = p(x), \text{ for } x \in \omega^\nu_{\underline{x}}, \text{ and any } p \in \mathcal{P}^k.$$

Using this formula and (3.66) with $\underline{y} = \underline{x}$, we obtain the estimate

$$\begin{aligned} \|u - \sum_{\underline{y}\in Q^\nu_{\underline{x}}} (\mathcal{A}^\nu_{\underline{x}}\, p^{\underline{x},\nu})(\underline{y})\, \phi^\nu_{\underline{y}}\|_{H^s(\omega^\nu_{\underline{x}})} &\leq \|u - p^{\underline{x},\nu}\|_{H^s(\omega^\nu_{\underline{x}})} \\ &\leq C(h^\nu_{\underline{x}})^{r^\nu_{\underline{x}}+1-s}\|u\|_{H^{r^\nu_{\underline{x}}+1}(B^{\underline{x}}_{\bar{\rho}h^\nu_{\underline{x}}})} \end{aligned} \tag{3.69}$$

for the first term.

Using assumption A6, we have

$$\|\phi^\nu_{\underline{y}}\|_{H^s(\omega^\nu_{\underline{z}})} \leq C(h^\nu_{\underline{y}})^{-s+n/2}, \text{ for all } \underline{y} \in Q^\nu_{\underline{x}}.$$

Thus

$$\begin{aligned} &\sum_{\underline{y}\in Q^\nu_{\underline{x}}} \big|(\mathcal{A}^\nu_{\underline{x}}\, p^{\underline{x},\nu})(\underline{y}) - (\mathcal{A}^\nu_{\underline{y}}\, p^{\underline{y},\nu})(\underline{y})\big|\, \|\phi^\nu_{\underline{y}}\|_{H^s(\omega^\nu_{\underline{x}})} \\ &\qquad \leq \sum_{\underline{y}\in Q^\nu_{\underline{x}}} (h^\nu_{\underline{y}})^{-s+n/2}\big|(\mathcal{A}^\nu_{\underline{x}}\, p^{\underline{x},\nu})(\underline{y}) - (\mathcal{A}^\nu_{\underline{y}}\, p^{\underline{y},\nu})(\underline{y})\big|. \end{aligned} \tag{3.70}$$

It remains to estimate the right-hand side of this inequality.

For $\underline{y} \in Q^\nu_{\underline{x}}$, we have $\omega^\nu_{\underline{y}} \in \Omega^\nu_{\underline{x}}$, and hence, using the assumption A4, $\omega^\nu_{\underline{y}} \in B^{\underline{x}}_{\bar{\rho}h^\nu_{\underline{x}}}$. Also $\omega^\nu_{\underline{y}} \in B^{\underline{y}}_{\bar{\rho}h^\nu_{\underline{y}}}$. Thus, using (3.66) with $s = 0$, we have

$$\begin{aligned} &\|p^{\underline{x},\nu} - p^{\underline{y},\nu}\|_{H^0(\omega^\nu_{\underline{y}})} \\ &\quad \leq \|p^{\underline{x},\nu} - u\|_{H^0(\omega^\nu_{\underline{y}})} + \|u - p^{\underline{y},\nu}\|_{H^0(\omega^\nu_{\underline{y}})} \\ &\quad \leq (h^\nu_{\underline{x}})^{r^\nu_{\underline{x}}+1}\|u\|_{H^{r^\nu_{\underline{x}}+1}(B^{\underline{x}}_{\bar{\rho}h^\nu_{\underline{x}}})} + (h^\nu_{\underline{y}})^{r^\nu_{\underline{y}}+1}\|u\|_{H^{r^\nu_{\underline{y}}+1}(B^{\underline{y}}_{\bar{\rho}h^\nu_{\underline{y}}})}. \end{aligned} \tag{3.71}$$

Now, using the assumption A7, we have

$$\begin{aligned} &\big|(\mathcal{A}^\nu_{\underline{x}}\, p^{\underline{x},\nu})(\underline{y}) - (\mathcal{A}^\nu_{\underline{y}}\, p^{\underline{y},\nu})(\underline{y})\big| \\ &\quad \leq \big|[(\mathcal{A}^\nu_{\underline{x}} - \mathcal{A}^\nu_{\underline{y}})p^{\underline{x},\nu}](\underline{y})\big| + \big|[\mathcal{A}^\nu_{\underline{y}}(p^{\underline{x},\nu} - p^{\underline{y},\nu})](\underline{y})\big| \\ &\quad \leq \|(\mathcal{A}^\nu_{\underline{x}} - \mathcal{A}^\nu_{\underline{y}})p^{\underline{x},\nu}\|_{L^\infty(\omega^\nu_{\underline{y}})} + \|\mathcal{A}^\nu_{\underline{y}}(p^{\underline{x},\nu} - p^{\underline{y},\nu})\|_{L^\infty(\omega^\nu_{\underline{y}})} \\ &\quad \leq C(h^\nu_{\underline{y}})^{-n/2}\big\{\|(\mathcal{A}^\nu_{\underline{x}} - \mathcal{A}^\nu_{\underline{y}})p^{\underline{x},\nu}\|_{H^0(\omega^\nu_{\underline{y}})} \\ &\qquad\qquad + \|\mathcal{A}^\nu_{\underline{y}}(p^{\underline{x},\nu} - p^{\underline{y},\nu})\|_{H^0(\omega^\nu_{\underline{y}})}\big\}. \end{aligned} \tag{3.72}$$

Also, using the assumption A5 and (3.71), we obtain

$$\begin{aligned}
&\|\mathcal{A}^\nu_{\underline{y}}(p^{\underline{x},\nu} - p^{\underline{y},\nu})\|_{H^0(\omega^\nu_{\underline{y}})} \\
&\quad \le C\|p^{\underline{x},\nu} - p^{\underline{y},\nu}\|_{H^0(\omega^\nu_{\underline{y}})} \\
&\quad \le C\big\{(h^\nu_{\underline{x}})^{r^\nu_{\underline{x}}+1}\|u\|_{H^{r^\nu_{\underline{x}}+1}(B^{\underline{x}}_{\bar\rho h^\nu_{\underline{x}}})} + (h^\nu_{\underline{y}})^{r^\nu_{\underline{y}}+1}\|u\|_{H^{r^\nu_{\underline{y}}+1}(B^{\underline{y}}_{\bar\rho h^\nu_{\underline{y}}})}\big\}. \qquad (3.73)
\end{aligned}$$

Moreover, from (3.64), we have

$$\|(\mathcal{A}^\nu_{\underline{x}} - \mathcal{A}^\nu_{\underline{y}})p^{\underline{x},\nu}\|_{H^0(\omega^\nu_{\underline{y}})} \le (h^\nu_{\underline{x}})^{r^\nu_{\underline{x}}+1}\|p^{\underline{x},\nu}\|_{H^0(\omega^\nu_{\underline{y}})}, \qquad (3.74)$$

and from (3.66), with $\underline{y} = \underline{x}$, and recalling that $h^\nu_{\underline{x}} \le 1$, we get

$$\begin{aligned}
\|p^{\underline{x},\nu}\|_{H^0(\omega^\nu_{\underline{y}})} &\le \|p^{\underline{x},\nu} - u\|_{H^0(\omega^\nu_{\underline{y}})} + \|u\|_{H^0(\omega^\nu_{\underline{y}})} \\
&\le C(h^\nu_{\underline{x}})^{r^\nu_{\underline{x}}+1}\|u\|_{H^{r^\nu_{\underline{x}}+1}(B^{\underline{x}}_{\bar\rho h^\nu_{\underline{x}}})} + \|u\|_{H^0(\omega^\nu_{\underline{y}})} \\
&\le C\|u\|_{H^{r^\nu_{\underline{x}}+1}(B^{\underline{x}}_{\bar\rho h^\nu_{\underline{x}}})}. \qquad (3.75)
\end{aligned}$$

Combining (3.72)–(3.75), we have

$$\begin{aligned}
&\big|(\mathcal{A}^\nu_{\underline{x}}\, p^{\underline{x},\nu})(\underline{y}) - (\mathcal{A}^\nu_{\underline{y}}\, p^{\underline{y},\nu})(\underline{y})\big| \\
&\le C(h^\nu_{\underline{y}})^{-n/2}\big\{(h^\nu_{\underline{x}})^{r^\nu_{\underline{x}}+1}\|u\|_{H^{r^\nu_{\underline{x}}+1}(B^{\underline{x}}_{\bar\rho h^\nu_{\underline{x}}})} + (h^\nu_{\underline{y}})^{r^\nu_{\underline{y}}+1}\|u\|_{H^{r^\nu_{\underline{y}}+1}(B^{\underline{y}}_{\bar\rho h^\nu_{\underline{y}}})}\big\}.
\end{aligned} \qquad (3.76)$$

Then we combine (3.70), (3.76), (3.60), and assumption A2 (*cf.* (3.60)) to get

$$\begin{aligned}
&\sum_{\underline{y}\in Q^\nu_{\underline{x}}} \big|(\mathcal{A}^\nu_{\underline{x}}\, p^{\underline{x},\nu})(\underline{y}) - (\mathcal{A}^\nu_{\underline{y}}\, p^{\underline{y},\nu})(\underline{y})\big|\|\phi^\nu_{\underline{y}}\|_{H^s(\omega^\omega_{\underline{x}})} \\
&\quad \le C\Big\{\kappa(h^\nu_{\underline{x}})^{r^\nu_{\underline{x}}+1}\|u\|_{H^{r^\nu_{\underline{x}}+1}(B^{\underline{x}}_{\bar\rho h^\nu_{\underline{x}}})} + \sum_{\underline{y}\in Q^\nu_{\underline{x}}}(h^\nu_{\underline{y}})^{r^\nu_{\underline{y}}+1}\|u\|_{H^{r^\nu_{\underline{y}}+1}(B^{\underline{y}}_{\bar\rho h^\nu_{\underline{y}}})}\Big\} \\
&\quad \le C\sum_{\underline{y}\in Q^\nu_{\underline{x}}}(h^\nu_{\underline{y}})^{r^\nu_{\underline{y}}+1}\|u\|_{H^{r^\nu_{\underline{y}}+1}(B^{\underline{y}}_{\bar\rho h^\nu_{\underline{y}}})}. \qquad (3.77)
\end{aligned}$$

which is an estimate for the second term in (3.68). Thus, from (3.68), (3.69), and (3.77), we have

$$\|u - \sum_{\underline{y}\in Q^\nu_{\underline{x}}} w^\nu_{\underline{y}}\phi^\nu_{\underline{y}}\|_{H^s(\omega^\nu_{\underline{x}})} \le C\sum_{\underline{y}\in Q^\nu_{\underline{x}}}(h^\nu_{\underline{y}})^{r^\nu_{\underline{y}}+1}\|u\|_{H^{r^\nu_{\underline{y}}+1}(B^{\underline{y}}_{\bar\rho h^\nu_{\underline{y}}})}. \qquad (3.78)$$

4. It remains to estimate $\|u - \sum_{\underline{y}\in X^\nu} w^\nu_{\underline{y}}\phi^\nu_{\underline{y}}\|_{H^s(\mathbb{R}^n)}$. Using (3.78), which is

valid for all $\underline{x} \in X^\nu$, and assumptions A1, A2, A4, we obtain

$$\begin{aligned} \|u - \sum_{\underline{y}\in X^\nu} w^\nu_{\underline{y}} \phi^\nu_{\underline{y}}\|^2_{H^s(\mathbb{R}^n)} &\leq \sum_{\underline{x}\in X^\nu} \|u - \sum_{\underline{y}\in X^\nu} w^\nu_{\underline{y}} \phi^\nu_{\underline{y}}\|^2_{H^s(\omega^\nu_{\underline{x}})} \\ &\leq C\kappa \sum_{\underline{x}\in X^\nu} \sum_{\underline{y}\in Q^\nu_{\underline{x}}} (h^\nu_{\underline{y}})^{2(r^\nu_{\underline{y}}+1-s)} \|u\|^2_{H^{r_{\underline{y}}+1}(B^{\underline{y}}_{\bar{\rho} h^\nu_{\underline{y}}})} \\ &\leq C \sum_{\underline{y}\in X^\nu} (h^\nu_{\underline{y}})^{2(r^\nu_{\underline{y}}+1-s)} \|u\|^2_{H^{r_{\underline{y}}+1}(B^{\underline{y}}_{\bar{\rho} h^\nu_{\underline{y}}})}, \end{aligned} \tag{3.79}$$

which is (3.65). $\square$

It will be useful to state estimate (3.65) in Theorem 3.8 in certain alternate ways. Given a family of particle-shape function systems $\{\mathcal{M}^\nu\}_{\nu\in N}$ satisfying A1–A7, define

$$h^\nu = \sup_{\underline{x}\in X^\nu} h^\nu_{\underline{x}}, \quad \text{for each } \nu. \tag{3.80}$$

With this definition, from (3.65) we have

$$\|u - \sum_{\underline{y}\in X^\nu} w^\nu_{\underline{y}} \phi^\nu_{\underline{y}}\|_{H^s(\mathbb{R}^n)} \leq C\Big(\sum_{\underline{y}\in X^\nu} (h^\nu)^{2(r^\nu_{\underline{y}}+1-s)} \|u\|_{H^{r^\nu_{\underline{y}}+1}(B^\nu_{\bar{\rho} h^\nu_{\underline{y}}})}\Big)^{1/2}. \tag{3.81}$$

Now, if $r^\nu_{\underline{x}^\nu} = k'$, where $0 \leq k' \leq k$, for all $\underline{y} \in X^\nu$, for all ν, then (3.81) leads to the following result.

Theorem 3.9 *Suppose the family of particle-shape function systems $\{\mathcal{M}^\nu\}_{\nu\in N}$ satisfies A1–A7, (3.64), and in addition, suppose $h^\nu \leq 1$, for all ν. Suppose $\|u\|_{H^{k'+1}(\mathbb{R}^n)} < \infty$, where $0 \leq k' \leq k$. Then there are weights $w^\nu_{\underline{y}} \in \mathbb{R}$ such that*

$$\|u - \sum_{\underline{y}\in X^\nu} w^\nu_{\underline{y}} \phi^\nu_{\underline{y}}\|_{H^s(\mathbb{R}^n)} \leq C(h^\nu)^{k'+1-s} \|u\|_{H^{k'+1}(\mathbb{R}^n)}, \tag{3.82}$$

for $0 \leq s \leq \min(q, k'+1)$, where C is independent of u and ν.

We note that if $h^{\nu_1} < h^{\nu_2}$, $\nu_1, \nu_2 \in N$, then we would view $\mathcal{M}^{\nu_1}$ as a "refinement" of $\mathcal{M}^{\nu_2}$.

There is yet another way to state the estimate (3.82). Let $0 < h \leq 1$, and suppose $\{\mathcal{M}^\nu\}_{\nu\in N}$, a family of particle-shape function systems satisfying A1–A7, (3.64), and in addition,

$$h^\nu = \sup_{\underline{x}^\nu\in X^\nu} h^\nu_{\underline{x}^\nu} \leq h, \quad \text{for each } \nu. \tag{3.83}$$

We can now think of $\nu = \nu(h)$ as determined by h, although, of course, many particle-shape function systems satisfy (3.83). We can, in fact, think of having a one-to-one correspondence between ν and h. Thus we can regard h as the parameter and write a family of particle-shape function systems as

$$\{\mathcal{M}^h\}_{0<h\leq 1} = \{X^h, \{h^h_{\underline{x}}, \omega^h_{\underline{x}}, \phi^h_{\underline{x}}\}_{\underline{x}\in X^h}\}_{0<h\leq 1}$$

instead of $\{\mathcal{M}^\nu\}_{\nu\in N}$. With this understanding that $\nu = \nu(h)$, the estimate (3.82) can be written as

$$\|u - \sum_{\underline{y}\in X^h} w^h_{\underline{y}} \phi^h_{\underline{y}}\|_{H^s(\mathbb{R}^n)} \leq C h^{k+1-s} \|u\|_{H^{k+1}(\mathbb{R}^n)}. \tag{3.84}$$

We are naturally interested in having $h \searrow 0$, and hence in considering $\nu(h)$'s for which $h^{\nu(h)} \searrow 0$. More specifically, we will often consider a sequence $h_1 > h_2 > \cdots \searrow$, and the corresponding sequence of particle systems $\mathcal{M}^{\nu_1}$, $\mathcal{M}^{\nu_2}$, $\cdots$, where $\nu_l = \nu(h_l)$.

We remark that the estimate (3.65) is stronger than (3.82) and (3.84), in that (3.65) uses $h^\nu_{\underline{x}^\nu}$ instead of the larger h^ν, and (3.65) allows a more general regularity assumption on the function u. The viewpoint, outlined in this paragraph, is similar to the usual view of meshes in FEM.

For a given family of particle-shape function systems $\{\mathcal{M}^\nu\}_{\nu\in N}$, we defined the space $\mathbb{V}^{k,q}_\nu$ in (3.62). With h, $0 < h \leq 1$, as the parameter, *i.e.,* for a given family $\mathcal{M}^h$, $0 < h \leq 1$, we will use the space

$$\mathbb{V}^{k,q}_h \equiv \mathbb{V}^{k,q}_{\nu(h)} = \text{span}\ \{\phi^h_{\underline{x}} : \underline{x} \in X^h\}. \tag{3.85}$$

So far, we have discussed the approximation of a function u defined on $\mathbb{R}^n$, by particle shape functions. We now consider u defined on Ω, where Ω is bounded domain, with Lipschitz continuous boundary, in $\mathbb{R}^n$. We now show that functions in $\mathbb{V}^{k,q}_{\Omega,h}$, defined by

$$\mathbb{V}^{k,q}_{\Omega,h} = \ \text{span}\{\phi^h_{\underline{x}}\big|_\Omega : \phi^h_{\underline{x}} \in \mathbb{V}^{k,q}_h,\ \text{where}\ \eta^h_{\underline{x}} \cap \Omega \neq \emptyset\}, \tag{3.86}$$

provide accurate approximation of functions u, defined on Ω.

Theorem 3.10 *Suppose $\mathcal{M}^h$, $0 < h \leq 1$, is a family of particle shape function systems satisfying A1–A7 and (3.64). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary, and suppose $u \in H^{k'+1}(\Omega)$, where $0 \leq k' \leq k$. Then there are weights $w^h_{\underline{y}} \in \mathbb{R}$ such that*

$$\|u - \sum_{\underline{y}\in X^h} w^h_{\underline{y}} \phi^h_{\underline{y}}\|_{H^s(\Omega)} \leq C h^{k'+1-s} \|u\|_{H^{k'+1}(\Omega)}, \tag{3.87}$$

for $0 \leq s \leq \min(q, k'+1)$, where the constant C is independent of u and h.

The proof of this theorem is based on using (3.84) on the extension $\bar{u} = Eu$, and is similar to the proof of Theorem 3.5. We omit the proof of this theorem. We note that the approximation $\sum_{\underline{y}\in X^h} w^h_{\underline{y}} \phi^h_{\underline{y}}$, obtained in Theorem 3.10, is such that

$$\sum_{\underline{y}\in X^h} w^h_{\underline{y}} \phi^h_{\underline{y}}\Big|_\Omega \in \mathbb{V}^{k,q}_{\Omega,h}.$$

In Section 3.2, we reviewed the notion of (t, k^*)-regular system $S_h(\Omega)$. In the next theorem, we show that $\mathbb{V}^{k,q}_{\Omega,h}$ is a $(k+1, q)$-system.

Theorem 3.11 *Suppose $\mathcal{M}^h$, $0 < h \leq 1$, is a family of particle shape function systems satisfying A1–A7 and (3.64). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary. Then $\mathbb{V}^{k,q}_{\Omega,h}$ is a $(k+1, q)$-regular system, where k is the order of quasi-reproducing shape functions in $\mathcal{M}^h$.*

The proof of this theorem is similar to the proof of Theorem 3.6, and will be omitted.

Remark 3.10 The space $\mathbb{V}^{k,q}_{\Omega,h}$ satisfies the local assumption (LA).

Quasi-Uniform Particle-Shape function System:

We will call a family of particle-shape function systems $\{\mathcal{M}^h\}_{0<h\leq 1}$ quasi-uniform if there is a β, $1 < \beta < \infty$, such that

$$\beta^{-1} \leq \frac{h^h_{\underline{x}}}{h^h_{\underline{y}}} \leq \beta, \quad \text{for all } \underline{x}, \underline{y} \in X^h, \text{ for all } 0 < h \leq 1. \tag{3.88}$$

We note that (3.88) is equivalent to

$$\beta^{-1} \leq \frac{h}{h^h_{\underline{y}}} \leq \beta, \quad \text{for all } \underline{y} \in X^h, \text{ for all } 0 < h \leq 1, \tag{3.89}$$

where h is defined by (3.83).

Remark 3.11 We can also define *uniform* particle-shape function system by imposing the condition

$$h^h_{\underline{x}} = h^h_{\underline{y}}, \quad \text{for all } \underline{x}, \underline{y} \in X^h, \text{ for all } 0 < h \leq 1.$$

We note that the system with uniformly distributed particles and the associated shape functions as defined in Section 3.1 is uniform. But uniform particle shape function systems may have particles that are not uniformly distributed.

Consider a family of particle-shape function systems $\{\mathcal{M}^h\}_{0<h\leq 1}$ satisfying the assumptions A1–A7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and define $A^h_\Omega = \{\underline{x} \in X^h : \mathring{\eta}^h_{\underline{x}} \cap \Omega\}$. Suppose $\mathcal{M}^h$ satisfies the following additional assumptions:

- $\mathcal{M}^h$ is quasi-uniform, *i.e.,* (3.89) holds.
- For all $x \in A^h_\Omega$, there is a $\beta > 0$ such that, for $0 \leq s \leq q$,

$$\beta^{-1} h^{\frac{n}{2}-s} \leq \|\phi^h_{\underline{y}}\|_{H^s(\omega^h_{\underline{x}} \cap \Omega)} \leq \beta h^{\frac{n}{2}-s}, \text{ for all } \underline{y} \in Q^h_{\underline{x}}, \tag{3.90}$$

  where $Q^h_{\underline{x}} = \{\underline{y} \in X^h : \eta^h_{\underline{y}} \cap \omega^h_{\underline{x}} \neq \emptyset\}$ (*cf.* A4).

- For all $w_{\underline{y}} \in \mathbb{R}$, for $\underline{y} \in Q^h_{\underline{x}}$, and $\underline{x} \in A^h_\Omega$, there is $C > 0$, independent of $\underline{x}$, such that

$$h^{-s}\Big[\sum_{\underline{y}\in Q^h_{\underline{x}}} |w_{\underline{y}}|^2 h^n\Big]^{1/2} \le C\|\sum_{\underline{y}\in Q^h_{\underline{x}}} w_{\underline{y}}\phi^h_{\underline{y}}\|_{H^s(\omega^h_{\underline{x}}\cap\Omega)}, \text{ for } 0 \le s \le q. \quad (3.91)$$

Then the particle space $\mathbb{V}^{k,q}_{\Omega,h}$ satisfies the inverse assumption IA, introduced in Section 3.2. To see this, consider $\underline{x} \in A^h_\Omega$. Then, using (3.90) and (3.91), we have

$$\begin{aligned}
\|\sum_{\underline{y}\in Q^h_{\underline{x}}} w_{\underline{y}}\phi^h_{\underline{y}}\|_{H^q(\omega^h_{\underline{x}}\cap\Omega)} &\le \sum_{\underline{y}\in Q^h_{\underline{x}}} |w_{\underline{y}}|\, \|\phi^h_{\underline{y}}\|_{H^q(\omega^h_{\underline{x}}\cap\Omega)} \\
&\le Ch^{\frac{n}{2}-q}\Big(\sum_{\underline{y}\in Q^h_{\underline{x}}} |w_{\underline{y}}|^2\Big)^{1/2} \\
&= Ch^{s-q}h^{-s}\Big(\sum_{\underline{y}\in Q^h_{\underline{x}}} |w_{\underline{y}}|^2 h^n\Big)^{1/2} \\
&\le Ch^{s-q}\|\sum_{\underline{y}\in Q^h_{\underline{x}}} w_{\underline{y}}\phi^h_{\underline{y}}\|_{H^s(\omega^h_{\underline{x}}\cap\Omega)}, \qquad (3.92)
\end{aligned}$$

where C depends on κ (*cf.* A2). Thus

$$\begin{aligned}
\|\sum_{\underline{y}\in A^h_\Omega} w_{\underline{y}}\phi^h_{\underline{y}}\|^2_{H^q(\Omega)} &\le \sum_{\underline{x}\in A^h_\Omega} \|\sum_{\underline{y}\in Q^h_{\underline{x}}} w_{\underline{y}}\phi^h_{\underline{y}}\|^2_{H^q(\omega^h_{\underline{x}}\cap\Omega)} \\
&\le Ch^{s-q}\sum_{\underline{x}\in A^h_\Omega} \|\sum_{\underline{y}\in Q^h_{\underline{x}}} w_{\underline{y}}\phi^h_{\underline{y}}\|^2_{H^s(\omega^h_{\underline{x}}\cap\Omega)} \\
&\le Ch^{s-q}\|\sum_{\underline{y}\in A^h_\Omega} w_{\underline{y}}\phi^h_{\underline{y}}\|^2_{H^s(\omega^h_{\underline{x}}\cap\Omega)}.
\end{aligned}$$

Since any element g of $\mathbb{V}^{k,q}_{\Omega,h}$ is of the form $\sum_{\underline{y}\in A^h_\Omega} w_{\underline{y}}\phi^h_{\underline{y}}\big|_\Omega$, we have shown that $\mathbb{V}^{k,q}_{\Omega,h}$ satisfies the inverse assumption IA. We summarize the above discussion in the following theorem:

Theorem 3.12 *Suppose $\mathcal{M}^h$, $0 < h \le 1$, is a family of quasi-uniform particle-shape function systems satisfying A1–A7, (3.90), and (3.91). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary. Then $\mathbb{V}^{k,q}_{\Omega,h}$ satisfies the inverse assumption IA.*

Remark 3.12 We can show that the particle space $\mathbb{V}^{k,q}_h$ also satisfies the inverse assumption IA, if $\mathcal{M}^h$, satisfying A1–A7, also satisfies (3.90) and (3.91) with $\Omega = \mathbb{R}^n$.

4 Construction and Selection of Particle Shape Functions

In Section 3, we presented an abstract description of particle-shape function systems with respect to uniform as well as non-uniform distribution of particles. We showed that if these particle-shape function systems satisfy certain properties (assumptions A1–A7 and (3.64)), they will have good approximation properties. In this section we will present an example of a particular particle-shape function system, where the shape functions are reproducing of order k, and show that under certain conditions they satisfy assumptions A1–A7, and hence have good approximation properties. We note that (3.64) is trivially satisfied. This example will also show that a wide variety of particle shape functions can be constructed. Therefore it is important to address the issue of selecting an appropriate class of shape functions that would yield efficient approximation of the solution of a particular problem, or a class of problems. We also present an interpolation result that will indicate a procedure for choosing a class of shape functions, among a given collection of such classes. Such shape functions will yield the smallest value of the usual Sobolev norm interpolation error, when the interpolated function is included in a higher order Sobolev space.

4.1 An Example of a Class of Particle Shape Functions

Several particle shape functions have been developed over the last decade. SPH shape functions [39] were introduced in the context of fluid dynamics, whereas Shepard functions [77] and MLS shape functions [53] were introduced in the context of data fitting with respect to irregularly distributed particles in higher dimensions. In the 90's, RKP shape functions were introduced [58] in the context of approximation of solutions of partial differential equations. In this paper, we describe the construction of RKP shape functions for non-uniform as well as uniform distribution of particles, and relate them to the abstract setting given in Section 3. Specifically, we will show that the resulting particle-shape function system satisfies assumptions A1–A7.

Non-uniformly distributed particles:

For $\nu \in N$, N an index set, let $X^\nu = \{x_i^\nu\}_{i\in\mathbb{Z}}$ where $x_i^\nu \in \mathbb{R}^n$. With each $x_i^\nu \in X^\nu$, we associate a positive number h_i^ν. We consider a fixed ν and often suppress the superscript ν, *e.g.,* we write x_i and h_i instead of x_i^ν and h_i^ν, respectively. We will comment about ν when appropriate.

Let $w(x) \geq 0$ be a continuous function with compact support, specifically,

$$\eta \equiv \text{ supp } w(x) = \overline{B}_R(0), \quad R > 0. \tag{4.1}$$

The function $w(x)$ is called a weight function (or window function).

The commonly used weight functions in 1-d are as follows:

(a) Gaussian:

$$w(x) = \begin{cases} \dfrac{e^{\delta(x/R)^2} - e^{\delta}}{1 - e^{\delta}}, & |x| \leq R \\ 0, & |x| \geq R, \end{cases} \tag{4.2}$$

where $\delta > 0$.

(b) Cubic spline:

$$w(x) = \begin{cases} \frac{2}{3} - 4(x/R)^2 + 4(x/R)^3, & |x| \leq R/2 \\ \frac{4}{3} - 4(x/R) + 4(x/R)^2 - \frac{4}{3}(x/R)^3, & R/2 \leq |x| \leq R \\ 0, & |x| > R. \end{cases} \tag{4.3}$$

(c) Conical:

$$w(x) = \begin{cases} [1 - (x/R)^2]^l, & |x| \leq R \\ 0, & |x| > R, \end{cases} \tag{4.4}$$

where $l = 1, 2 \ldots$. We note that one may consider non-symmetric versions of some of these weight functions, as was done in [2].

In $\mathbb{R}^n$, $w(x)$ can be constructed from 1-d weight function $w(x)$ (symmetric) as $w(x) = w(\|x\|)$, where $\|x\|$ is the Euclidean length of x. $w(x)$ can also be constructed as, $w(x) = \prod_{j=1}^{n} w({}_jx)$, where $x = ({}_1x, {}_2x, \ldots, {}_nx) \in \mathbb{R}^n$. Consequently, η will be an $n-$cube. However, we will assume η given by (4.1) in this section.

For each j, we define

$$w_j(x) = w\left(\frac{x - x_j}{h_j}\right). \tag{4.5}$$

Clearly,

$$\eta_j \equiv \text{ supp } w_j(x) = \overline{B}_{Rh_j}(x_j). \tag{4.6}$$

Let

$$Q_i = \{x_j : \mathring{\eta}_i \cap \mathring{\eta}_j \neq \emptyset\}, \tag{4.7}$$

and assume that

$$\cup_{j \in \mathbb{Z}} \mathring{\eta}_j = \mathbb{R}^n, \tag{4.8}$$

$$\text{card } Q_i \leq \kappa, \text{ for all } i \in \mathbb{Z}, \tag{4.9}$$

where κ is independent of i and ν.

For a given integer k, $k \geq 0$, the RKP shape function $\phi_j(x)$, associated with the particle x_j, is defined by

$$\phi_j(x) = w_j(x) \sum_{|\alpha| \leq k} (x - x_j)^{\alpha} b_{\alpha}(x), \tag{4.10}$$

where $b_{\alpha}(x)$ are chosen so that

$$\sum_{j \in \mathbb{Z}} p(x_j)\phi_j(x) = p(x), \text{ for } x \in \mathbb{R}^n, \text{ for } p \in \mathcal{P}^k(\mathbb{R}^n), \tag{4.11}$$

so that $\{\phi_j(x)\}_{j\in\mathbb{Z}}$ are reproducing of order k. This gives rise to a linear system in $b_\alpha(x)$; namely

$$\sum_{|\alpha|\le k} m_{\alpha+\beta}(x)b_\alpha(x) = \delta_{|\beta|,0}, \text{ for } |\beta| \le k, \tag{4.12}$$

where $\delta_{|\beta|,|\alpha|}$ is the Kronecker delta, and

$$m_\alpha(x) = \sum_{j\in\mathbb{Z}} w_j(x)(x-x_j)^\alpha.$$

It is clear from (4.6) and (4.10) that

$$\text{supp } \phi_j(x) = \text{ supp } w_j(x) = \eta_j. \tag{4.13}$$

We now briefly, describe the derivation of (4.12). For a fixed $y \in \mathbb{R}^n$, consider

$$p_\beta(x) = (y-x)^\beta, \quad 0 \le |\beta| \le k.$$

Using $p(x) = p_\beta(x)$ in (4.11) we get

$$\sum_{j\in\mathbb{Z}} (y-x_j)^\beta \phi_j(x) = (y-x)^\beta,$$

and letting $y = x$ in the above equality, we have

$$\sum_{j\in\mathbb{Z}} (x-x_j)^\beta \phi_j(x) = \delta_{|\beta|,0}, \quad 0 \le |\beta| \le k. \tag{4.14}$$

Thus (4.11) implies (4.14). In fact, one can also show that (4.14) implies (4.11). Now using (4.10) in (4.14), we get

$$\begin{aligned}
\delta_{|\beta|,0} &= \sum_{j\in\mathbb{Z}} (x-x_j)^\beta \phi_j(x) \\
&= \sum_{j\in\mathbb{Z}} (x-x_j)^\beta w_j(x) \sum_{|\alpha|\le k} (x-x_j)^\alpha b_\alpha(x) \\
&= \sum_{|\alpha|\le k} b_\alpha(x) \sum_{j\in\mathbb{Z}} w_j(x)(x-x_j)^{\alpha+\beta} \\
&= \sum_{|\alpha|\le k} m_{\alpha+\beta}(x) b_\alpha(x),
\end{aligned} \tag{4.15}$$

which is (4.12).

We now consider the unique solvability of (4.11). For $k = 0$, the linear system (4.12) is $m_0(x)b_0(x) = \big[\sum_{i\in\mathbb{Z}} w_i(x)\big]b_0(x) = 1$. Assuming $\sum_{i\in\mathbb{Z}} w_i(x) \ne 0,\ x \in \mathbb{R}^n$, we have $b_0(x) = 1/m_0(x)$. Therefore from (4.10), we have

$$\phi_j(x) = \frac{w_j(x)}{\sum_{i\in\mathbb{Z}} w_i(x)}, \quad j \in \mathbb{Z}.$$

This expression for $\{\phi_j(x)\}$ gives another verification that they form a partition of unity. These shape function are called Shepard functions; they were introduced in [77].

The unique solvability of (4.12), for $k \geq 1$, depends on the weight functions w_j's and on the distribution of the particles $\{x_i\}$ in $\mathbb{R}^n$. The required distribution of particles in turn is related to the interpolation problem in $\mathbb{R}^n$. It was shown in [48] that a necessary condition for unique solvability of (4.12) is that, for $x \in \mathbb{R}^n$,

$$\text{card } A(x) \geq \ \dim \mathcal{P}^k, \tag{4.16}$$

where

$$A(x) = \{x_l : x \in \eta_l\}. \tag{4.17}$$

For $k = 1$, in was shown in [48] that the linear system (4.12) is non-singular if the following conditions are satisfied:

(a) There are constants $C_1, C_2 > 0$ independent of ν, and $h > 0$, such that

$$C_1 \leq \frac{h_i}{h} \leq C_2, \text{ for all } i \in \mathbb{Z}; \tag{4.18}$$

(b) There are constants $\tilde{C}_1, \tilde{C}_2 > 0$, independent of ν, such that for any $x \in \mathbb{R}^n$, there are $(n+1)$ particles $x_{i_l} \in A(x), l = 0, \ldots, n$, such that

$$\min_{0 \leq l \leq n} w\left(\frac{x - x_{i_l}}{h}\right) \geq \tilde{C}_1 > 0 \tag{4.19}$$

and

$$\text{Volume } K(x_{i_0}, x_{i_1}, \ldots, x_{i_n}) \geq \tilde{C}_2 h^n, \tag{4.20}$$

where $K(x_{i_0}, x_{i_1}, \ldots, x_{i_n})$ is the simplex with vertices $x_{i_l}, l = 0, 1, \ldots, n$.

We will now cast RKP shape functions, discussed above, in the framework of a particle-shape function system, introduced in Section 3.3. We started with a collection of particles $X^\nu = \{x_j^\nu\}_{j \in \mathbb{Z}}$, where $x_j^\nu \in \mathbb{R}^n$, and positive numbers h_j^ν. Corresponding to each particle $x_j^\nu \in X^\nu$, we associated, in (4.10), the RKP shape function, $\phi_j^\nu = \phi_j$ with compact support $\eta_j^\nu = \eta_j = \overline{B}_{Rh_j}(x_j)$, where the parameter R was related to the weight function $w(x)$. It was shown in [48] that if $w(x) \in C^q(\mathbb{R}^n)$, then $\phi_j \in C^q(\mathbb{R}^n)$, and thus $\phi_j \in H^q(\mathbb{R}^n)$; here we assume $q = 1$. We recall that the conditions (4.8), (4.8), (4.16), (4.18)–(4.20) were required for the construction of shape functions, ϕ_j, $j \in \mathbb{Z}$. We let $\omega_j^\nu = \omega_j \equiv \mathring{\eta}_j$; certainly ω_j^ν's are bounded domains. We now show that the family of particle-shape function systems $\{\mathcal{M}^\nu\}_{\nu \in N}$, where

$$\mathcal{M}^\nu = \Big\{X^\nu, \{h_i^\nu, \omega_i^\nu, \phi_i^\nu\}\Big\},$$

with these choices for ϕ_i^ν and ω_i^ν, satisfies assumptions A1–A7 in Section 3.3. We will continue to use the notation, introduced earlier in this section, and suppress ν; the statements of A1–A7 using this notation should be clear.

- Since $\omega_i = \mathring{\eta}_i$ for $i \in \mathbb{Z}$, assumption A1 follows from (4.8). We also see that the sets $S^\nu_{\underline{x}} \equiv S_i$ and $Q^\nu_{\underline{x}} \equiv Q_i$, introduced in assumptions A2 and A4 are same. Thus A2 follows from (4.9).

- Assumption A3 is immediate from the definition ω_i.

- Since $\omega_j = \mathring{\eta}_j$, the set $\Omega^\nu_{\underline{x}}$, introduced in assumption A4, is given by $\Omega^\nu_{\underline{x}} = \Omega_i = \cup_{x_j \in Q_i} \mathring{\eta}_j$. Since η_j's are balls of radius Rh_j, it is easily seen, using (4.18), that assumption A4 is satisfied with $\bar{\rho} = 3RC_2/C_1$.

- RKP shape functions, ϕ_j, considered here, are reproducing of order $k = 1$, *i.e.,* they satisfy (4.11) with $k = 1$. Thus A5 is satisfied with $\mathcal{A}^\nu_{\underline{x}} = \mathcal{A}_i = I$ (identity), for all $i \in \mathbb{Z}$, with $k = 1$.

- It was shown in [48] that if the weight function $w(x) \in C^q$, then

  $$\|\phi_i\|_{W^{s,\infty}(\mathring{\eta}_i)} \leq C(h_i)^{-s}, \quad \text{for } 0 \leq s \leq q, \text{ for } i \in \mathbb{Z}.$$

  Thus using a scaling argument and this estimate, we obtain

  $$\|\phi_i\|_{H^s(\mathring{\eta}_i)} \leq C(h_i)^{-s+n/2}, \quad \text{for } 0 \leq s \leq q, \text{ for } i \in \mathbb{Z}, \tag{4.21}$$

  where h and h_i satisfy (4.18). Recall that we assumed $q = 1$. Now let $x_j \in Q_i$. Then

  $$\|\phi_j\|_{H^s(\mathring{\eta}_i)} = \|\phi_j\|_{H^s(\mathring{\eta}_i \cap \mathring{\eta}_j)} \leq \|\phi_j\|_{H^s(\mathring{\eta}_j)},$$

  and combining this with (4.21), we get, for $0 \leq s \leq 1$,

  $$\|\phi_j\|_{H^s(\mathring{\eta}_i)} \leq C(h_i)^{-s+n/2}, \text{ for all } x_j \in Q_i, \text{ for all } i \in \mathbb{Z},$$

  which is assumption A6 with $q = 1$.

- A scaling argument shows that assumption A7 is satisfied.

We remark that (3.61) of Remark 3.4, together with condition (b) (following (4.18)), establishes a lower bound of κ, namely $(n+1) \leq \kappa$.

We have thus shown that assumptions A1–A7, with $k = 1$ and $q = 1$, are satisfied by RKP particle-shape function systems provided (4.8), (4.9), (4.16), (4.18)–(4.20) are satisfied. Thus we can apply Theorem 3.8 to obtain an approximation error estimate for RKP particle-shape function systems. Note that the condition (3.64) in Theorem 3.8 is trivially satisfied with $\mathcal{A}^\nu_{\underline{x}} = I$ for all $\underline{x} \in X^\nu$. We remark that an interpolation error estimate, under the assumptions (4.8), (4.9), (4.16), (4.18)–(4.20), was also obtained in [48].

We note that A1–A7 only guarantee good approximability of the shape functions; they do not provide a recipe to construct particle shape functions that are quasi-reproducing or reproducing of order k. In fact the availability of such particle shape functions is assumed in A5. Further assumptions may be needed to construct such shape functions; for example (4.16), (4.18)–(4.20) were needed to construct RKP particle shape functions. Therefore, there should be enough restrictions on the particle distributions and the supports of shape functions so that it is possible to construct these shape functions satisfying A1–A7, thereby ensuring good approximation properties.

Uniformly distributed particles:

We consider the uniformly distributed particles $x_j^h = jh$, $j \in \mathbb{Z}^n$ as in Section 3.2. This is a special case of the non-uniformly distributed particles considered in the first part of this section. For each x_j^h, we define $w_j^h(x) = w(\frac{x-x_i^h}{h})$, where $w(x) \geq 0$ is a continuous weight function with compact support $\eta = \overline{B}_R(0)$. Clearly, $\eta_j^h \equiv \text{supp } w_j^h(x) = \overline{B}_{Rh}(x_j^h)$. It can be easily shown that if $R = 3\sqrt{n}/2$ (in fact, we need only $R > \sqrt{n}$), then (4.8), (4.9) with $\kappa = (4R+1)^n$, (4.18) with $C_1 = C_2 = 1$, and (4.20) with $\tilde{C}_2 = 1/2$ are satisfied. If $w(x) = w(r)$, with $r = \|x\|$, is monotonically decreasing in r, then it also can be easily shown that (4.19) is satisfied with $\tilde{C}_1 = w(\sqrt{n})$. Therefore, RKP shape functions $\phi_i^h(x)$, associated with x_i^h, for all $i \in \mathbb{Z}^n$ can be constructed using the procedure described in (4.10), (4.11) and (4.12) for $k = 1$, namely

$$\phi_j^h(x) = w_j^h(x) \sum_{|\alpha| \leq k} (x - x_j^h)^\alpha b_\alpha^h(x), \tag{4.22}$$

where $\{b_\alpha^h(x)\}_{|\alpha| \leq k}$ is the solution of

$$\sum_{|\alpha| \leq k} m_{\alpha+\beta}^h(x) b_\alpha^h(x) = \delta_{|\beta|,0}, \quad |\beta| \leq k, \tag{4.23}$$

with $k = 1$, and

$$m_\alpha^h(x) = \sum_{j \in \mathbb{Z}^n} w_j^h(x)(x - x_j^h)^\alpha. \tag{4.24}$$

Shape functions ϕ_j^h's satisfy

$$\sum_{j \in \mathbb{Z}^n} p(x_j^h)\phi_j^h(x) = p(x), \text{ for all } x \in \mathbb{R}^n, \text{ for all } p \in \mathcal{P}^k(\mathbb{R}^n). \tag{4.25}$$

As with the non-uniformly distributed particles, we consider the family of particle-shape function systems

$$\mathcal{M}^h = \left\{ X^h, \{h_{\underline{x}}^h, \omega_{\underline{x}}^h, \phi_{\underline{x}}^h\}_{\underline{x} \in X^h} \right\}, \quad 0 < h \leq 1,$$

for RKP shape functions with respect to uniformly distributed particles, by letting $X^h = \{\underline{x} = x_j^h : j \in \mathbb{Z}^n\}$ and using $h_{\underline{x}}^h = h$, $\omega_{\underline{x}}^h = \eta_j^h$ and $\phi_{\underline{x}}^h = \phi_j^h$. Note that here we used the parameter h instead of ν. We have shown above that conditions (4.8), (4.9), (4.18)–(4.20) are satisfied, with $w(x) = w(r)$, a monotonically decreasing weight function in r, and $R = 3\sqrt{n}/2$. Therefore, based on the discussion on RKP particle-shape function systems for non-uniformly distributed particles, it clear that $\{\mathcal{M}^h\}_{0<h\leq 1}$ satisfies the assumptions A1–A7 with $k = 1$, ensuring good approximation properties of the RKP shape functions.

We recall that in Section 3.1, the particle shape function $\phi_i^h(x)$ was defined in (3.1) by scaling and translating the basic shape function $\phi(x)$ for uniformly distributed particles, *i.e.,* they were translation invariant. We will show that the RKP shape functions $\{\phi_i^h\}_{i\in\mathbb{Z}^n}$, constructed via (4.22) and (4.23), also satisfy (3.1) with $\phi(x) = \phi_0^1(x)$ (*i.e.,* with $i = 0$ and $h = 1$). We assume that the linear system (4.23) is invertible for $k \geq 1$.

From (4.22) and (4.23) with $i = 0$ and $h = 1$, we have

$$\phi(x) = w(x) \sum_{|\alpha|\leq k} x^\alpha b_\alpha^1(x), \tag{4.26}$$

where $b_\alpha^1(x)$ are the solutions of

$$\sum_{|\alpha|\leq k} m_{\alpha+\beta}^1(x) b_\alpha^1(x) = \delta_{|\beta|,0}, \text{ for } |\beta| \leq k, \tag{4.27}$$

and

$$m_\alpha^1(x) = \sum_{j\in\mathbb{Z}^n} w(x-j)(x-j)^\alpha. \tag{4.28}$$

We replace x by $\frac{x-x_i^h}{h}$ in (4.27) and (4.28) to get

$$\sum_{|\alpha|\leq k} m_{\alpha+\beta}^1\left(\frac{x-x_i^h}{h}\right) b_\alpha\left(\frac{x-x_i^h}{h}\right) = \delta_{|\beta|,0}, \text{ for } |\beta| \leq k, \tag{4.29}$$

where

$$\begin{aligned} m_\alpha^1\left(\frac{x-x_i^h}{h}\right) &= \sum_{j\in\mathbb{Z}^n} w\left(\frac{x-x_i^h}{h} - j\right)\left(\frac{x-x_i^h}{h} - j\right)^\alpha \\ &= \sum_{j\in\mathbb{Z}^n} w\left(\frac{x-x_{i+j}^h}{h}\right)\left(\frac{x-x_{i+j}^h}{h}\right)^\alpha \\ &= \frac{1}{h^{|\alpha|}} \sum_{j\in\mathbb{Z}^n} w_j^h(x)(x-x_j^h)^\alpha \\ &= \frac{1}{h^{|\alpha|}} m_\alpha^h(x). \end{aligned} \tag{4.30}$$

Using (4.30) in (4.29), we get

$$\sum_{|\alpha|\leq k} \frac{1}{h^{|\alpha+\beta|}} m_{\alpha+\beta}^h(x) b_\alpha\left(\frac{x-x_i^h}{h}\right) = \delta_{|\beta|,0},$$

and therefore,

$$\sum_{|\alpha|\leq k} m_{\alpha+\beta}^h(x) \frac{1}{h^{|\alpha|}} b_\alpha\left(\frac{x-x_i^h}{h}\right) = h^{|\beta|}\delta_{|\beta|,0} = \delta_{|\beta|,0}, \text{ for all } |\beta| \leq k. \tag{4.31}$$

Since $b_\alpha^h(x)$'s are unique solutions of (4.23), it is clear from (4.31) that

$$b_\alpha^h(x) = \frac{1}{h^{|\alpha|}} b_\alpha(\frac{x - x_i^h}{h}),$$

and thus from (4.26), we have

$$\begin{aligned}
\phi(\frac{x - x_i^h}{h}) &= w(\frac{x - x_i^h}{h}) \sum_{|\alpha| \le k} (\frac{x - x_i^h}{h})^\alpha b_\alpha(\frac{x - x_i^h}{h}) \\
&= w_i^h(x) \sum_{|\alpha| \le k} (x - x_i^h)^\alpha \frac{1}{h^{|\alpha|}} b_\alpha(\frac{x - x_i^h}{h}) \\
&= w_i^h(x) \sum_{|\alpha| \le k} (x - x_i^h)^\alpha b_\alpha^h(x) \\
&= \phi_i^h(x).
\end{aligned}$$

Thus, for uniformly distributed particles, RKP shape functions satisfy (3.1), *i.e.,* they are translation invariant.

Remark 4.1 To approximate functions defined on a bounded domain Ω, we use the restrictions of RKP shape functions on Ω, as described in Section 3.3 (*cf.* (3.86) and Theorem 3.10). We note that the RKP shape functions corresponding to the particles near the boundary of Ω, as defined here, are different from the RKP shape functions defined in [48] and [58]. But they are same for particles inside Ω, sufficiently away from the boundary $\partial\Omega$. They are also same when $\Omega = \mathbb{R}^n$.

4.2 Interpolation and Selection

In this section, we will address the interpolation of a function in terms of particle shape function, and will propose a procedure to select shape function that will yield efficient approximation. We consider uniformly distributed particles $\{x_j^h\}$ in $\mathbb{R}^n$, and the associated particle shape functions $\{\phi_j^h\}$, defined in (3.1), where $\phi \in H^q(\mathbb{R}^n)$ with $q \ge 1$ has compact support; supp $\phi \subset B_R(0)$. We have seen that ϕ_j^h's are translation invariant, supp $\phi_j^h \in B_{Rh}(x_j^h)$, and in addition they satisfy

$$\|\phi_j^h\|_{H^1(\mathbb{R}^n)} \le h^{n/2-1} \|\phi\|_{H^1(\mathbb{R}^n)}. \tag{4.32}$$

We assume that $\{\phi_j^h\}$ are reproducing of order k, *i.e.,* (4.25) holds.

Let Ω be a bounded domain in $\mathbb{R}^n$. We will consider a smooth function $u(x)$ defined in Ω and study the error $u - \tilde{\mathcal{I}}_h u$, where $\tilde{\mathcal{I}}_h u$ is the "interpolant" of u in terms of ϕ_j^h. The results in this subsection are from [13], and we refer to [13] for some of the details that we do not present here.

We now define the "interpolant" $\tilde{\mathcal{I}}_h u$ of a function u. For any $x \in \mathbb{R}^n$, let

$$A_x^h = \{k \in \mathbb{Z}^n : x \in \mathring{\eta}_k^h\}$$

A_x^h is called the influence set for the point x. Then $(\tilde{\mathcal{I}}_h u)(x)$ is defined as

$$(\tilde{\mathcal{I}}_h u)(x) = \sum_{j \in A_x^h} u(x_j^h)\phi_j^h(x). \tag{4.33}$$

In (4.33), we, of course, assume that $u(x_j^h)$ is defined for all $j \in A_x^h$. If $p \in \mathcal{P}^k$, then from (4.25) we have

$$\sum_{j \in A_x^h} p(x_j^h)\phi_j^h(x) = \sum_{j \in \mathbb{Z}^n} u(x_j^h)\phi_j^h(x) = p(x), \text{ for all } x \in \mathbb{R}^n, \tag{4.34}$$

i.e., $\tilde{\mathcal{I}}_h p = p$. Now let $u \in H^s(\Omega)$ with $s > n/2$. For some $x \in \Omega$, the particles x_j^h for $j \in A_x^h$ may be outside Ω, and $u(x_j^h)$ may not be defined. To define $\tilde{\mathcal{I}}_h u(x)$ for $u \in H^s(\Omega)$ and for all $x \in \Omega$, we need an extension $\bar{u}$ of u in a ball B_{R_0} containing Ω such that $\operatorname{dist}(\partial\Omega, \partial B_{R_0}) > \rho h$, and $u \in H^s(B_{R_0})$. Then,

$$(\tilde{\mathcal{I}}_h u)(x) \equiv (\tilde{\mathcal{I}}_h \bar{u})(x) \quad = \quad \sum_{j \in A_x^h} \bar{u}(x_j^h)\phi_j^h(x), \text{ for } x \in \Omega, \tag{4.35}$$

is well defined. For an extension $\bar{u}$, we may use $\bar{u} = Eu$, where Eu was defined in (3.51). Thus, $(\tilde{\mathcal{I}}_h u)(x)$ for $x \in \Omega$ will depend on few values of $\bar{u}(x_j^h)$, where the particle x_j^h is just outside Ω. We remark that $\tilde{\mathcal{I}}_h u$ is not an interpolant of u in the usual sense, since, generally, $\phi_j^h(x_i^h) \neq \delta_{ij}$, and hence $(\tilde{\mathcal{I}}_h u)(x_j^h) \neq u(x_j^h)$.

We define the function

$$\xi_\alpha^h(x) = x^\alpha - \sum_{i \in A_x^h} (x_i^h)^\alpha \phi_i^h(x), \ |\alpha| = k+1, \text{ for } x \in \mathbb{R}^n. \tag{4.36}$$

We will also use

$$\xi_\alpha(x) \equiv \xi_\alpha^1(x) = x^\alpha - \sum_{i \in A_x^1} i^\alpha \phi(x - i), \ |\alpha| = k+1, \text{ for } x \in \mathbb{R}^n, \tag{4.37}$$

where A_x^1 is A_x^h with $h = 1$. These functions will play an important role in the analysis presented in this subsection, as well as in Section 5. We note that $\xi_\alpha(x)$ is the error between the polynomial x^α, with $|\alpha| = k+1$, and its interpolant when $h = 1$. In 1-d, we will write these functions as $\xi_{k+1}^h(x)$ and $\xi_{k+1}(x)$ respectively.

We begin with certain results about these functions. We first present some notations that will be used in these results. Let I_j^h be the *cell* centered at x_j^h, defined by

$$I_j^h = \{x : \|x - x_j^h\|_\infty \equiv \max_{i=1,\ldots,n} |x_i - x_{j_i}| \leq h/2\}.$$

For each I_j^h, we define

$$A_j^h = \{k \in \mathbb{Z}^n : \mathring{\eta}_k^h \cap I_j^h \neq \emptyset\},$$

and

$$B_j^h = \{\cup_{k\in A_j^h} B_{Rh}(x_k^h)\} \cup I_j^h.$$

We note that cardinality of A_j^h is finite, and is bounded independent of j and h. Also there exists $\bar{R} > 0$, independent of j and h, such that $B_j^h \subset \tilde{B}_j^h \equiv B_{\bar{R}h}(x_j^h)$ and $\cup_{j\in\mathbb{Z}^n} \tilde{B}_j^h = \mathbb{R}^n$.

Lemma 4.1 *([13]) $\xi_\alpha^h(x)$, with $|\alpha| = k+1$, is periodic, i.e.,*

$$\xi_\alpha^h(x + x_j^h) = \xi_\alpha^h(x), \quad \textit{for any } x_j^h. \tag{4.38}$$

Proof. We first note that

$$(x + x_j^h)^\alpha = x^\alpha + p(x; x_j^h), \tag{4.39}$$

where $p(x; x_j^h)$ is a polynomial in x of degree $\le k$ with coefficients that depend on x_j^h. Now using (4.39), with $x = x_i^h$, and the fact that the ϕ_i^h's are translation invariant and reproducing of order k, we get

$$\begin{aligned}
\sum_{i\in\mathbb{Z}^n} (x_i^h)^\alpha \phi_i^h(x + x_j^h) &= \sum_{i\in\mathbb{Z}^n} (x_i^h)^\alpha \phi_{i-j}^h(x) \\
&= \sum_{i\in\mathbb{Z}^n} (x_{i+j}^h)^\alpha \phi_i^h(x) \\
&= \sum_{i\in\mathbb{Z}^n} (x_i^h + x_j^h)^\alpha \phi_i^h(x) \\
&= \sum_{i\in\mathbb{Z}^n} (x_i^h)^\alpha \phi_i^h(x) + \sum_{i\in\mathbb{Z}^n} p(x_i^h; x_j^h)\phi_i^h(x) \\
&= \sum_{i\in\mathbb{Z}^n} (x_i^h)^\alpha \phi_i^h(x) + p(x, x_j^h).
\end{aligned} \tag{4.40}$$

From from (4.36), (4.39) and (4.40), we get

$$\xi_\alpha^h(x + x_j^h) = x^\alpha - \sum_{i\in\mathbb{Z}^n} (x_i^h)^\alpha \phi_i^h(x) = \xi_\alpha^h(x),$$

which is the desired result. □

Lemma 4.2 *([13]) Let $\alpha = \alpha(i)$, $i = 1, \cdots, M_k$ be an enumeration of the multi-indices α with $|\alpha(i)| = k+1$. Let I_j^h be the cell centered at the particle x_j^h. Then, for $d_\alpha \in \mathbb{R}$, we have*

$$\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \xi_\alpha^h(x) \|_{H^1(I_j^h)}^2 = h^{2k+n} \mathbb{V}^T (A + h^2 B)\mathbb{V}, \tag{4.41}$$

where $\mathbb{V} = [d_{\alpha(1)}, d_{\alpha(2)}, \ldots, d_{\alpha(M_k)}]^T$ *and* A, B *are* $M_k \times M_k$ *matrices given by*

$$A_{lm} = \int_I \frac{1}{\alpha(l)!\alpha(m)!} \nabla \xi_{\alpha(l)} \cdot \nabla \xi_{\alpha(m)} \, dx, \tag{4.42}$$

$$B_{lm} = \int_I \frac{1}{\alpha(l)!\alpha(m)!} \xi_{\alpha(l)} \xi_{\alpha(m)} \, dx \;, \tag{4.43}$$

respectively, and $I = [-1/2, 1/2]^n$*.*

Note: The matrices A and B are independent of I_j^h.

Proof. A simple scaling argument, used with (3.1), shows that

$$\xi_\alpha\left(\frac{x}{h}\right) = h^{-(k+1)} \xi_\alpha^h(x).$$

Now, using the periodicity of $\xi_\alpha^h(x)$, a standard scaling argument, and this identity, we have

$$\begin{aligned} \| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \nabla \xi_\alpha^h(x) \|^2_{H^0(I_j^h)} &= \| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \nabla \xi_\alpha^h(x) \|^2_{H^0(I_0^h)} \\ &= h^{2(k+1)} \| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \nabla [\xi_\alpha\left(\frac{x}{h}\right)] \|^2_{H^0(I_0^h)} \\ &= h^{2(k+1)} h^{n-2} \| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \nabla \xi_\alpha(y) \|^2_{H^0(I)} \\ &= h^{2k+n} \mathbb{V}^T A \mathbb{V}. \end{aligned} \tag{4.44}$$

Using a similar argument, we have

$$\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \xi_\alpha^h(x) \|^2_{H^0(I_j^h)} = h^{2k+2+n} \mathbb{V}^T B \mathbb{V}.$$

Combining this identity with (4.44), we get

$$\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \xi_\alpha^h(x) \|^2_{H^1(I_j^h)} = h^{2k+n} \mathbb{V}^T (A + h^2 B) \mathbb{V},$$

which is the desired result. □

Lemma 4.3 *([13]) Let* I_j^h *be the cell centered at the particle* x_j^h*, and consider the corresponding set* $\tilde{B}_j^h$*. Suppose* $u \in H^{k+2+q}(\tilde{B}_j^h)$ *with* $q > \frac{n}{2}$ *when* $n \geq 2$*, and* $q = 0$ *when* $n = 1$*. Then,*

(a) for any $\delta > 0$*,*

$$\begin{aligned} \|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(I_j^h)} \leq\; & (1+\delta^2) \| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} (D^\alpha u)(x_j^h) \xi_\alpha^h(x) \|^2_{H^1(I_j^h)} \\ & + (1 + \frac{1}{\delta^2}) C h^{2k+2} \sum_{|\alpha|=k+2} \|D^\alpha u\|^2_{H^q(\tilde{B}_j^h)}. \end{aligned} \tag{4.45}$$

and

(b) for any $\delta > 0$,

$$\begin{aligned}\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)\xi_\alpha^h(x)\|^2_{H^1(I_j^h)} &\leq (1+\delta^2)\|u-\tilde{\mathcal{I}}_h u\|^2_{H^1(I_j^h)} \\ &\quad +(1+\frac{1}{\delta^2})Ch^{2k+2}\sum_{|\alpha|=k+2}\|D^\alpha u\|^2_{H^q(\tilde{B}_j^h)}.\end{aligned} \tag{4.46}$$

The proof of this result is based on Taylor's Theorem, a bound on the remainder in Taylor's Theorem, and a bound on the interpolant of the same remainder. We do not include the proof here, and refer to [13].

We will now study the interpolation error $u-\tilde{\mathcal{I}}_h u$, where u is a smooth function in Ω. An interpolation error estimate, namely $\|u-\tilde{\mathcal{I}}_h u\|_{H^1(\Omega)} \approx O(h^k)$, was proved in [48, 59] for the RKP shape functions. A similar order of convergence in the $H^{1,\infty}$ norm was also obtained for MLS shape functions in [1, 2]. We note that the definitions of $\tilde{\mathcal{I}}_h u$ for the RKP shape functions and MLS shape functions, presented in these papers, are slightly different from our definition as given in (4.35). From the proof of the our next result, we will obtain an estimate of $\|u-\tilde{\mathcal{I}}_h u\|_{H^1(\Omega)}$ where the shape functions shape functions are reproducing of order k. Moreover, this theorem gives some information on the size of $\frac{\|u-\tilde{\mathcal{I}}_h u\|_{H^1(\Omega)}}{h^k}$, which facilitates the selection of "good" shape functions, which will be discussed later.

We now present the main result of this section. We define certain sets, which will be used in this result:

$$\begin{aligned} \bar{\mathcal{A}}^h &= \{k \in \mathbb{Z}^n : \Omega \cap \mathring{I}_k^h \neq \emptyset\}, & \bar{\Omega}_h &= \cup_{j\in\bar{\mathcal{A}}^h} I_j^h, \\ \underline{\mathcal{A}}^h &= \{k \in \mathbb{Z}^n : I_k^h \subset \Omega\}, & \underline{\Omega}_h &= \cup_{j\in\underline{\mathcal{A}}^h} I_j^h, \\ \underline{B}^h &= \{\cup_{j\in\underline{\mathcal{A}}^h}\tilde{B}_j^h\} \cup \Omega, & \bar{B}^h &= \cup_{j\in\bar{\mathcal{A}}^h}\tilde{B}_j^h. \end{aligned}$$

It is clear that $\underline{\Omega}_h \subset \Omega \subset \bar{\Omega}_h$, and $|\Omega - \underline{\Omega}_h| \to 0$, $|\bar{\Omega}_h - \Omega| \to 0$ as $h \to 0$. Also $\Omega \subset \underline{B}^h \subset \bar{B}^h$, and $|\underline{B}^h - \Omega| \to 0$, $|\bar{B}^h - \Omega| \to 0$ as $h \to 0$.

Theorem 4.1 *([13]) Let $\bar{\lambda}$ be the largest eigenvalue of the matrix A given in (4.42). Suppose $q > \frac{n}{2}$ when $n \geq 2$, and $q = 0$ when $n = 1$. Then, we have*

$$\sup_{u\in H^{k+2+q}(\Omega)} \lim_{h\to 0} \frac{\|u-\tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k}Q_h(u)} = \bar{\lambda}, \tag{4.47}$$

where

$$Q_h(u) = |u|^2_{H^{k+1}(\Omega)} + h\sum_{|\alpha|=k+2}\|D^\alpha u\|^2_{H^q(\Omega)}. \tag{4.48}$$

Note: In (4.47), we consider $u \in H^{k+2+q}(\Omega)$ such that $u \notin \mathcal{P}^k$.

Proof. We will first prove that for $u \in H^{k+2+q}(\Omega)$,

$$\lim_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k} Q_h(u)} = \frac{\int_\Omega V^T(x) A V(x)\, dx}{|u|^2_{H^{k+1}(\Omega)}}, \tag{4.49}$$

where

$$V^T(x) = [D^{\alpha(1)}u(x), D^{\alpha(2)}u(x), \ldots, D^{\alpha(M_k)}u(x)],$$

and $\alpha(i)$, $1 \le i \le M_k$, are the multi-indices with $|\alpha(i)| = k+1$.

Let $u \in H^{k+2+q}(\Omega)$, and suppose $\bar{u}$ is an extension of u, as discussed before. Since, $\Omega \subset \bar{\Omega}_h$, we have

$$\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)} \le \|\bar{u} - \tilde{\mathcal{I}}_h \bar{u}\|^2_{H^1(\bar{\Omega}_h)} = \sum_{j\in\bar{\mathcal{A}}^h} \|\bar{u} - \tilde{\mathcal{I}}_h \bar{u}\|^2_{H^1(I^h_j)}.$$

Therefore, using (4.45), (4.41), and recalling that $\bar{B}^h = \cup_{j\in\bar{\mathcal{A}}^h} \tilde{B}^h_j$, we get for any $\delta > 0$,

$$\begin{aligned}
\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)} &\le (1+\delta^2) \sum_{j\in\bar{\mathcal{A}}^h} \| \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha \bar{u})(x^h_j)\xi^h_\alpha(x)\|^2_{H^1(I^h_j)} \\
&\quad + (1+\frac{1}{\delta^2}) C h^{2k+2} \sum_{j\in\bar{\mathcal{A}}^h} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|^2_{H^q(\tilde{B}^h_j)} \\
&\le (1+\delta^2) h^{2k} \sum_{j\in\bar{\mathcal{A}}^h} h^n V_j^T (A + h^2 B) V_j \\
&\quad + (1+\frac{1}{\delta^2}) C h^{2k+2} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|^2_{H^q(\bar{B}^h)},
\end{aligned} \tag{4.50}$$

where

$$V_j^T = [D^{\alpha(1)}\bar{u}(x^h_j), D^{\alpha(2)}\bar{u}(x^h_j), \ldots, D^{\alpha(M_k)}\bar{u}(x^h_j)].$$

Therefore, dividing (4.50) by $h^{2k}Q_h(u)$, where $Q_h(u)$ is defined in (4.48), we get

$$\begin{aligned}
\frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k} Q_h(u)} &\le (1+\delta^2) \frac{\sum_{j\in\bar{\mathcal{A}}^h} h^n V_j^T (A + h^2 B) V_j}{Q_h(u)} \\
&+ (1+\frac{1}{\delta^2}) C h^2 \frac{\sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|^2_{H^q(\bar{B}^h)}}{Q_h(u)}.
\end{aligned} \tag{4.51}$$

A typical term of the quadratic form $V_j^T(A + h^2 B)V_j$ is

$$D^{\alpha(i)}\bar{u}(x^h_j)(A_{il} + h^2 B_{il}) D^{\alpha(l)}\bar{u}(x^h_j).$$

Since

$$\lim_{h\to 0} \sum_{j\in\bar{\mathcal{A}}^h} h^n D^{\alpha(i)}\bar{u}(x^h_j) A_{il} D^{\alpha(l)}\bar{u}(x^h_j) = \int_\Omega D^{\alpha(i)}u(x) A_{il} D^{\alpha(l)}u(x)\, dx$$

and

$$\lim_{h\to 0} h^2 \sum_{j\in\bar{\mathcal{A}}^h} h^n D^{\alpha(i)}\bar{u}(x_j^h) B_{il} D^{\alpha(l)}\bar{u}(x_j^h) = 0,$$

we have

$$\lim_{h\to 0} \sum_{j\in\bar{\mathcal{A}}^h} h^n V_j^T (A + h^2 B) V_j = \int_\Omega V^T(x) A V(x)\, dx. \tag{4.52}$$

Since $|\bar{B}^h - \Omega| \to 0$ as $h \to 0$, we have

$$\lim_{h\to 0} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|^2_{H^q(\bar{B}^h)} = \sum_{|\alpha|=k+2} \|D^\alpha u\|^2_{H^q(\Omega)}. \tag{4.53}$$

Also $\lim_{h\to 0} Q_h(u) = |u|_{H^{k+1}(\Omega)}$. Thus, for any $\delta > 0$, using (4.52) and (4.53) in (4.51), we get

$$\limsup_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k} Q_h(u)} \le (1+\delta^2) \frac{\int_\Omega V^T(x) A V(x)\, dx}{|u|^2_{H^{k+1}(\Omega)}},$$

and, since $\delta > 0$ is arbitrary, we have

$$\limsup_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k} Q_h(u)} \le \frac{\int_\Omega V^T(x) A V(x)\, dx}{|u|^2_{H^{k+1}(\Omega)}}. \tag{4.54}$$

Following the argument leading to (4.54), but using $\underline{\mathcal{A}}^h$, $\underline{B}^h$, and (4.46) instead of $\bar{\mathcal{A}}^h$, $\bar{B}^h$, and (4.45), respectively, we can also show that

$$\frac{\int_\Omega V^T(x) A V(x)\, dx}{|u|^2_{H^{k+1}(\Omega)}} \le \liminf_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k} Q_h(u)}. \tag{4.55}$$

Combining (4.54) and (4.55), we see that $\lim_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k} Q_h(u)}$ exists, and

$$\lim_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k} Q_h(u)} = \frac{\int_\Omega V^T(x) A V(x)\, dx}{|u|^2_{H^{k+1}(\Omega)}},$$

which is (4.49).

Since $\bar{\lambda}$ is the largest eigenvalue of the matrix A, from the usual variational characterization of eigenvalues, we have

$$\int_\Omega V^T(x) A V(x)\, dx \le \bar{\lambda} \int_\Omega \sum_{i=1}^{M_k} |D^{\alpha(i)} u(x)|^2\, dx = \bar{\lambda} |u|^2_{H^{k+1}(\Omega)}.$$

Thus from (4.49) we get

$$\lim_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k} Q_h(u)} \le \bar{\lambda}, \quad \text{for any } u \in H^{k+2+q}(\Omega).$$

Hence

$$\sup_{u\in H^{p+2+q}(\Omega)} \lim_{h\to 0} \frac{\|u-\tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k}Q_h(u)} \le \bar{\lambda}. \tag{4.56}$$

Let $\bar{v} = [v_1, v_2, \cdots, v_{M_k}]^T$ be an eigenvector of A corresponding to $\bar{\lambda}$. Then it is easily seen that there is a $u \in \mathcal{P}^{k+1}$ such that the vector $V(x) = \bar{v}$. For this particular u, we have

$$\frac{\int_\Omega V^T(x)AV(x)\,dx}{|u|^2_{H^{k+1}(\Omega)}} = \bar{\lambda}.$$

Hence, from (4.56) we conclude that

$$\sup_{u\in H^{k+2+q}(\Omega)} \lim_{h\to 0} \frac{\|u-\tilde{\mathcal{I}}_h u\|^2_{H^1(\Omega)}}{h^{2k}Q_h(u)} = \bar{\lambda},$$

which is the desired result. □

Remark 4.2 We know from (4.35) that the interpolant of a smooth function depends on its extension to $\mathbb{R}^n$. But it is clear from the proof of Theorem 4.1 that (4.47) is valid for any extension satisfying (3.51).

Remark 4.3 We note that same result holds for the H^1-seminorm of the interpolation error, *i.e.,* for $q > \frac{n}{2}$ when $n \ge 2$, and $q = 0$ when $n = 1$, we have

$$\sup_{u\in H^{k+2+q}(\Omega)} \lim_{h\to 0} \frac{|u-\tilde{\mathcal{I}}_h u|^2_{H^1(\Omega)}}{h^{2k}[|u|^2_{H^{k+1}(\Omega)} + h\sum_{|\alpha|=k+2}\|D^\alpha u\|^2_{H^q(\Omega)}]} = \bar{\lambda}$$

Remark 4.4 From (4.51) in the proof of Theorem 4.1, we can obtain an interpolation error estimate,

$$\|u-\tilde{\mathcal{I}}_h u\|_{H^1(\Omega)} \le Ch^k\|u\|_{H^{k+2+q}(\Omega)},$$

where C may depend on Ω, but is independent of u and h. We note however, that this is not the optimal error estimate. For an outline of the proof, see [13].

We have seen in Remark 4.4 that if the particle shape functions are reproducing of order k, then for a smooth function u,

$$\|u-\tilde{\mathcal{I}}_h u\|_{H^1(\Omega)} \approx O(h^k),$$

where $\tilde{\mathcal{I}}_h u$ is the interpolation of u as defined in (4.35). There are many classes of shape functions that have these properties. We have seen in Section 4.1 that translation invariant RKP shape functions depend on the weight function $w(x)$, and different choices of $w(x)$ will generate different classes of such shape functions.

We will assess the aproximability of a family $\{\phi_j^h\}$ of shape functions by the size of $\bar{\lambda}$, the largest eigenvalue of the matrix A defined in (4.42). We note that $\bar{\lambda}$ is computable, and depends only on the basic shape function $\phi(x)$. We emphasize that $\bar{\lambda}$ does not depend on u or on h. From (4.47), we know that

$$\frac{\|u - \tilde{\mathcal{I}}_h u\|_{H^1(\Omega)}}{h^p \sqrt{Q_h(u)}} \lessapprox \sqrt{\bar{\lambda}}, \quad \text{for small } h.$$

Thus we see that $\bar{\lambda}$ is a useful measure of the approximabilty of the family $\{\phi_j^h\}$, determined from the basic shape function $\phi(x)$.

We will illustrate our selection scheme in 1-d, and will rank the shape functions according to to their approximability. We note that in 1-d, $\bar{\lambda} = (\frac{|\xi_{k+1}|_{H^1(0,1)}}{(k+1)!})^2$. In the rest of this paper, we will suppress $H^1(0,1)$ in $|\xi_{k+1}|_{H^1(0,1)}$, and instead write $|\xi_{k+1}|_1$.

We considered four different classes of RKP shape functions, reproducing of order 1, corresponding to four different weight functions $w(x)$. These $w(x)$'s were (4.2) with $\delta = 2$, (4.3), and (4.4) with $l = 2, 4$. We then computed $|\xi_{k+1}|_1$ for each of these four classes of shape functions for $R = 1.7$; we obtained

$$|\xi_{k+1}|_1 = \begin{cases} 0.237, & \text{for } w(x) \text{ in (4.4), } l = 2 \\ 0.203, & \text{for } w(x) \text{ in (4.2), } \delta = 2 \\ 0.095, & \text{for } w(x) \text{ in (4.3)} \\ 0.029, & \text{for } w(x) \text{ in (4.4),} l = 4 \end{cases}$$

We choose the RKP shape functions corresponding to $w(x)$ given in (4.4) with $l = 4$, since these shape functions yield the smallest value of $|\xi_{k+1}|_1$. We note that the value of $|\xi_{k+1}|_1$ depends strongly on R, and the shape function corresponding to $w(x)$ given in (4.4) with $l = 4$ may not be our choice for other values of R. We refer to [13] for futher discussion on this issue.

To validate our criterion of selection of the shape functions, we have considered the function $u(x) = x^4$ on the interval $\Omega = (0, 1)$ and computed the error $|u - \tilde{\mathcal{I}}_h u|_{H^1(\Omega)}$. $\tilde{\mathcal{I}}_h u$ is the interpolant of u with respect to the four classes of RKP shape functions described in the last paragraph, with $h = 1/n$, $n = 40, 50, \ldots, 100$. We note that the definition of $\tilde{\mathcal{I}}_h u$ requires the values of $u(x)$ in a small neighborhood of Ω, and we have extended $u = x^4$ outside Ω by itself. We summarize the results in the following table.

n	$\vert u-\tilde{\mathcal{I}}_h u\vert_{H^1(\Omega)}$			
	Conical: $l=2$	Gauss: $\delta=2$	Cubic Spline	Conical: $l=4$
40	1.607e-2	1.376e-2	6.435e-3	2.283e-3
50	1.281e-2	1.096e-2	5.130e-3	1.730e-3
60	1.066e-2	9.112e-3	4.267e-3	1.396e-3
70	9.126e-3	7.800e-3	3.653e-3	1.172e-3
80	7.980e-3	6.819e-3	3.194e-3	1.012e-3
90	7.090e-3	6.058e-3	2.838e-3	8.908e-4
100	6.379e-3	5.449e-3	2.553e-3	7.962e-4

Table 4.1: The H^1-seminorm of the error, $|u-\tilde{\mathcal{I}}_h u|_{H^1(\Omega)}$, where $\tilde{\mathcal{I}}_h u$ is the interpolant of $u(x)=x^4$ using RKP shape functions that are reproducing of order 1, corresponding to different weight functions $w(x)$. The radius of support of $\omega(x)$ is $R=1.7$.

From Table 4.1, it is clear that the error $|u-\tilde{\mathcal{I}}_h u|_{H^1(\Omega)}$ can be ranked according to the size of $|\xi_2|_1$ for the four choices of $\omega(x)$ considered here with $R=1.7$; the error and $|\xi_2|_1$ are both minimal when $w(x)$ is the conical weight function with $l=4$.

This selection scheme is based on (4.47), and we know from Remark 4.2 that (4.47) is valid for any extension. We refer to [13] for an experimental illustration of this fact. We remark that this selection scheme is also valid for the projection error, which will be indicated by our results in the next section.

5 Superconvergence of the gradient of the solution in L_2

Superconvergence is an important feature of finite element methods, which allows an accurate approximation of the derivatives of the solution of the underlying BVP. In this section, we will discuss the idea of superconvergence when particle shape functions are used to approximate the solution of a BVP. We will consider uniformly distributed particles and the associated particle shape functions, which were developed in Sections 3.1 and 3.2. For uniformly distributed particles, a careful analysis in 1-d can be easily generalized to higher dimensions. Thus, in this section, we present the results in 1-d, but by restricting our analysis to 1-d, we avoid some details that arise in higher dimensional analysis.

We will use the notation that was introduced in Section 3.1, but restricted to 1-d, *i.e.,* for $h>0$, we consider $x_j^h = jh$, $j\in\mathbb{Z}$, and the corresponding shape function ϕ_j^h defined in (3.1). We assume that the shape functions are reproducing of order k. We use the following notation:

$$I_j^h = (x_j^h, x_{j+1}^h), \quad A_j^h = \{m\in\mathbb{Z} : \eta_m^h \cap I_j^h \neq \emptyset\};$$
$$I_j = (j, j+1), \quad A_j \equiv A_j^1 \;\; (\text{with } h=1).$$

We assume that

$$\operatorname{card}(A_j) \le \kappa,$$

or equivalently,

$$\operatorname{card}(A_j^h) \le \kappa,$$

where κ is independent of j and h. We assume that the basic shape function $\phi(x)$ is such that, for any $v(x) = \sum_{i\in\mathbb{Z}} c_i\phi_i(x)$ for $x \in I_0$, there exist positive constants $C_1,\ C_2$, independent of v, but may depend on κ, such that

$$C_1 \sum_{j\in A_0} c_i^2 \le \int_{I_0} v^2\,dx \le C_2 \sum_{j\in A_0} c_i^2. \tag{5.1}$$

This implies that the functions $\{\phi_i(x)\}_{i\in A_0}$ are linearly independent in I_0, *i.e.,*

$$\sum_{j\in A_0} c_j\phi_j(x) = 0,\ x \in I_0 \text{ implies } c_j = 0, j \in A_0.$$

Throughout this section, we use $C,\ C_1,\ C_2$ as generic constants, which will have different values in different places.

Consider $\Omega = (-c, d) \subset \mathbb{R}$. Let $u_0 \in H^1(\Omega)$ be the solution of the Neumann problem

$$B(u_0, v) = \mathcal{F}(v), \quad \text{for all } v \in H^1(\Omega) \tag{5.2}$$

where

$$B(u, v) = \int_\Omega (u'v' + uv)\,dx \quad \text{and} \quad \mathcal{F}(v) = \int_\Omega fv\,dx$$

as in (2.4) and (2.5). We will often use the notation $B^F(u, v)$ to denote the above bilinear form, where the Ω is replaced by another domain F.

Let $u_h \in V_{\Omega,h}^{k,q}$ be the solution of

$$B(u_h, v) = \mathcal{F}(v), \quad \text{for all } v \in V_{\Omega,h}^{k,q}, \tag{5.3}$$

where $V_{\Omega,h}^{k,q}$ was defined in (3.55). It is clear from(5.2) and (5.3) that

$$B(u_0 - u_h, v) = 0, \quad \text{for all } v \in V_{\Omega,h}^{k,q}, \tag{5.4}$$

and we easily have

$$\|u_h\|_{H^1(\Omega)} \le \|u_0\|_{H^1(\Omega)}. \tag{5.5}$$

Recall that the functions in $V_{\Omega,h}^{k,q}$ are restrictions of the functions in $S_h \equiv V_h^{k,q}$ on Ω (*cf.* (3.2) and (3.55)). Thus (5.4) is true when $V_{\Omega,h}^{k,q}$ is replaced by S_h.

We assume that for any $\rho > 0$,

$$\|u_0 - u_h\|_{L_2(B_\rho(0))} \le Ch^{k+1}\|u_0\|_{H^{k+1}(\Omega)}\rho^{\frac{1}{2}}, \tag{5.6}$$

and there are positive constants C_1, C_2, independent of u, h, and ρ, such that

$$C_1 h^k \rho^{\frac{1}{2}} \le \frac{\|u_0' - u_h'\|_{L_2(B_\rho(0))}}{\|u_0\|_{H^{k+1}(\Omega)}} \le C_2 h^k \rho^{\frac{1}{2}}, \tag{5.7}$$

where $B_\rho(0) = \{x : |x| < \rho\}$ and $B_\rho(0) \subset \Omega$. We will write $B_\rho \equiv B_\rho(0)$ through out this section.

The main goal of this section is to investigate the error $u'(x) - u_h'(x)$ in a neighborhood of $x = 0$, *i.e.,* for $x \in B_H \subset\subset \Omega$ and $H = h^\gamma$, $\gamma < 1$, where γ will be chosen later. We will prove the following result:

Theorem 5.1 *Suppose u_0 and u_h satisfy (5.6) and (5.7), and let $e_h = u_0 - u_h$. Moreover, assume that $u_0 \in W_\infty^{k+2}(B_{2H})$. Then for h small enough, there exists $\epsilon^* > 0$, such that*

$$\frac{\|e_h' - T(u_0)\xi_{k+1}^{h\,\prime}\|_{L_2(B_H)}}{\|e_h'\|_{L_2(B_{2H})}} \leq Ch^{\epsilon^*}$$

where $T(u_0) = \frac{u_0^{(k+1)}(0)}{(k+1)!}$ and $\xi_{k+1}^h(x) = h^{k+1}\xi_{k+1}(\frac{x}{h})$; ξ_{k+1} is defined in (4.37).

Remark 5.1 Theorem 5.1 is a superconvergence result. It shows that

$$\|e_h' - T(u_0)\xi_{k+1}^{h\,\prime}\|_{L_2(B_H)} << \|e_h'\|_{L_2(B_H)}.$$

This allows one, for example, to analyze the effectiveness of an error estimator as was done in [19].

Since the all the results in this paper have been presented in terms of L_2 based norms (*i.e.,* in terms of the usual Sobolev norms), we also present this result in terms of L_2 based norm. Superconvergence in L_∞ will be addressed in a forthcoming paper. Assuming superconvergence in L_∞, the superconvergence points and superconvergence recoveries in the case of particle shape functions can be obtained analogously as in [19]. At the end of this section, we will see an example where the superconvergence points are distributed differently than in the classical FEM.

Remark 5.2 The essential aspects of superconvergence analysis in the classical FEM are interior estimates, developed in [71], [75], [88]. This analysis strongly utilizes the polynomial character of the shape functions. Here, in the case of particle shape functions, we had to develop another approach to the analysis of superconvergence, which is based on weighted Sololev spaces. The main idea of the proof of our superconvergence result is to show that *locally,* the approximation error is asymptotically same as the error in the interpolation of a polynomial of degree $k+1$ by particle shape functions. The analysis is technical; we present the main idea of this analysis in this section.

Remark 5.3 Assumptions (5.6) and (5.7) are directly related to the control of pollution, as in FEM. The assumption that $u_0 \in W_\infty^{k+1}(B_\rho)$ is analogous to the assumption in FEM (see [19]).

To prove Theorem 5.1, we will first develop certain ideas and establish many technical results. Towards this end, for given parameters $H = h^\gamma$, with $\gamma < 1$, and $\alpha \geq 1$, we define the function $g(x)$ by

$$g(x) = \begin{cases} 1, & -H \leq x \leq H \\ e^{-\alpha(x-H)}, & x > H \\ e^{\alpha(H+x)}, & x < -H. \end{cases} \tag{5.8}$$

where α is such that $\alpha h < 1$, and will be chosen later. We note that a proper choice of γ and α is crucial for the analysis presented in this section. Often, we will use $g \equiv g(x)$, $g_i \equiv g(x_i^h)$ and $g_{i+\frac{1}{2}} \equiv g(x_i^h + h/2)$.

Generalized Interpolant and certain norm estimates:

We first introduce the idea of generalized interpolant of a function u, which is different than $\tilde{\mathcal{I}}_h u$ defined in Section 4.2. Let $\tilde{I}_0 \equiv I_{-1} \cup I_0 \cup \{0\} = (-1, 1)$ and $\tilde{A}_0 \equiv A_{-1} \cup A_0$. Then from (5.1), it is clear that there are positive constants C_1, C_2, independent of $v = \sum_{i\in\mathbb{Z}} c_i\phi_i(x)$, but may depend on κ, such that

$$C_1 \sum_{j\in\tilde{A}_0} c_i^2 \leq \int_{\tilde{I}_0} v^2\, dx \leq C_2 \sum_{j\in\tilde{A}_0} c_i^2, \tag{5.9}$$

which implies that $\{\phi_i(x)\}_{i\in\tilde{A}_0}$ are also linearly independent in $\tilde{I}_0$. We define $\psi_0(x) = \sum_{i\in\tilde{A}_0} a_i\phi_i(x)$ with supp $\psi_0 = \bar{\tilde{I}}_0$ (closure of $\tilde{I}_0$), such that

$$\begin{aligned} \int_{\tilde{I}_0} \psi_0(x)\phi_0(x)\, dx &= 1, \\ \int_{\tilde{I}_0} \psi_0(x)\phi_j(x)\, dx &= 0, \text{ for all } j \in \tilde{A}_0,\ j \neq 0. \end{aligned} \tag{5.10}$$

Using (5.9), we can show that

$$\|\psi_0\|_{L_2(\tilde{I}_0)} \leq C. \tag{5.11}$$

We also note that, since $\{\phi_i(x)\}_{i\in\tilde{A}_0}$ form a partition unity on $\tilde{I}_0$, from (5.10) we have

$$\int_{\tilde{I}_0} \psi_0(x)\, dx = \int_{\tilde{I}_0} \psi_0(x) \sum_{i\in\tilde{A}_0} \phi_i(x)\, dx = \int_{\tilde{I}_0} \psi_0(x)\phi_0(x)\, dx = 1. \tag{5.12}$$

Let $\psi_i^h(x) = \psi_0(\frac{x}{h} - i)$. Then supp $\psi_i^h = \overline{\tilde{I}_i^h}$, where $\tilde{I}_i^h = (x_{i-1}^h, x_{i+1}^h)$. Note that $\cup_{i\in Z}\tilde{I}_i^h = \mathbb{R}$. Now for a given $v \in L_2(\mathbb{R})$. we define the *generalized interpolant* of v as

$$\tilde{\mathcal{I}}_h^* v(x) = \sum_{i\in\mathbb{Z}} \Psi_i^h(v)\phi_i^h(x), \tag{5.13}$$

where

$$\Psi_i^h(v) = \frac{1}{h}\int_{\tilde{I}_i^h} \psi_i^h(x)v(x)\,dx. \tag{5.14}$$

We note that $\tilde{\mathcal{I}}_h^* v(x)$ depends on the $v(y)$ for $y \in \cup_{i\in A^h(x)} \tilde{I}_i^h$, where $A^h(x) = \{l \in \mathbb{Z} : x \in \eta_l^h\}$. We also define

$$\tilde{A}_i^h = \{m \in \mathbb{Z} : \eta_m^h \cap \tilde{I}_i^h \neq \emptyset\}.$$

Lemma 5.1 *Suppose* $v(x) = \sum_{i\in\mathbb{Z}} c_i^h \phi_i^h(x)$*. Then*

$$\begin{aligned} c_i^h &= \Psi_i^h(v) & (5.15)\\ \text{and } \tilde{\mathcal{I}}_h^* v(x) &= v(x). & (5.16) \end{aligned}$$

Proof. From (5.10) and the definition of $\Psi_i^h(v)$ in (5.14), $i \in \mathbb{Z}$, we have

$$\begin{aligned} \Psi_i^h(v) &= \frac{1}{h}\int_{\tilde{I}_i^h} \psi_i^h(x)v(x)\,dx \\ &= \frac{1}{h}\int_{\tilde{I}_i^h} \psi_0(\frac{x}{h}-i) \sum_{j\in\tilde{A}_i^h} c_j^h \phi(\frac{x}{h}-j)\,dx \\ &= \int_{\tilde{I}_0} \psi_0(y) \sum_{j\in\tilde{A}_i^h} c_j^h \phi_{j-i}(y)\,dy \\ &= c_i^h, \end{aligned}$$

which is (5.15). Now using (5.15) in (5.13). we get (5.16).□

Remark 5.4 We note that if v is a linear combination of ϕ_i^h's only locally, *i.e.,* in a bounded open interval, then $\tilde{\mathcal{I}}_h^* v = v$ only in the interior of that open interval. More precisely, $\tilde{\mathcal{I}}_h^* v = v$ in an interval I if v is a linear combination of ϕ_i's in $\cup_{x\in I} \cup_{i\in A^h(x)} \tilde{I}_i^h$.

We will use the following result later.

Lemma 5.2 *Let* Ω *be a bounded interval, and suppose* $u \in L_2(\Omega)$*. Then*

$$\|\tilde{\mathcal{I}}_h^* Eu\|_{H^1(\mathbb{R})} \leq Ch^{-1}\|u\|_{L_2(\Omega)}$$

where E *is the extension operator satisfying (3.51).*

Proof. We first note that the extension Eu of u satifies $\|Eu\|_{L_2(\mathbb{R})} \leq C\|u\|_{L_2(\Omega)}$. Now, from (5.13) and (4.32)

$$\begin{aligned} \|\tilde{\mathcal{I}}_h^* Eu\|^2_{H^1(\tilde{I}_i^h)} &\leq C \sum_{j\in\tilde{A}_i^h} |\Psi_j^h(Eu)|^2\ \|\phi_j^h\|^2_{H^1(\mathring{\eta}_j)} \\ &\leq Ch^{-1} \sum_{j\in\tilde{A}_i^h} |\Psi_j^h(Eu)|^2, & (5.17) \end{aligned}$$

where C depends on κ; and using Schwartz inequality on (5.14) with $v = Eu$, and a scaling argument, we get

$$
\begin{aligned}
|\Psi_j^h(Eu)|^2 &\leq \frac{1}{h^2}\left(\int_{\tilde{I}_j^h} \psi_j^h(x)Eu(x)\,dx\right)^2 \\
&\leq \frac{1}{h^2}[\int_{\tilde{I}_j^h}(\psi_j^h)^2\,dx][\int_{\tilde{I}_j^h}(Eu)^2\,dx] \\
&\leq \frac{1}{h}\|\psi_0\|^2_{L_2(\tilde{I}_0)}[\int_{\tilde{I}_j^h}(Eu)^2\,dx]. \qquad (5.18)
\end{aligned}
$$

Thus, from (5.17), (5.18), and the fact that $\|Eu\|_{L_2(\mathbb{R})} \leq C\|u\|_{L_2(\Omega)}$ we have

$$
\|\tilde{\mathcal{I}}_h^* Eu\|^2_{H^1(\mathbb{R})} \leq Ch^{-2}\|u\|^2_{L_2(\Omega)},
$$

which is the desired result. □

Remark 5.5 We can also show that

$$
\|\tilde{\mathcal{I}}_h^* Eu\|_{L_2(\mathbb{R})} \leq C\|u\|_{L_2(\Omega)}
$$

using the same arguments as in the proof of Lemma 5.2.

Consider the function $v(x) = \sum_{i\in A_j^h} c_i^h \phi_i^h(x)$ on I_j^h. Then, using scaling, translation, and (5.1), we have

$$
C_1 h \sum_{j\in A_j^h}(c_i^h)^2 \leq \int_{I_j^h} v^2\,dx \leq C_2 h \sum_{j\in A_j^h}(c_i^h)^2, \qquad (5.19)
$$

where $C_1,\ C_2$ are positive constants, independent of h and j, but may depend on κ. Using (5.19), we can show that if $v(x) = \sum_{i\in\mathbb{Z}} c_i^h \phi_i^h(x) = 0$ in L_2, then $c_i^h = 0$, for all $i \in \mathbb{Z}$, *i.e.*, $\{\phi_i^h\}$ are linearly independent.

We will now prove certain lower bounds for $\int_{I_j^h} gv^2\,dx$ and $\int_{I_j^h} g{v'}^2\,dx$, where $g(x)$ has been defined before. We first prove the following inequality.

Lemma 5.3 *Let $i_0,\ i_1$ be integers such that $i_0 < i_1$, and suppose $\{c_i\}_{i=i_0}^{i_1}$ are real numbers. Then there exists a positive constant C, depending only on $i_1 - i_0$, such that for any k, $i_0 \leq k \leq i_1$, we have*

$$
\sum_{i=i_0}^{i_1} g_{i+\frac{1}{2}}(c_i - c_k)^2 \leq C\sum_{i=i_0}^{i_1-1} g_{i+\frac{1}{2}}(c_{i+1} - c_i)^2. \qquad (5.20)
$$

Proof. Suppose the integers i_0, i_1 are such that $H < i_0 h < i_1 h$, where $H = h^\gamma,\ \gamma < 1$. Then

$$
\sum_{i=i_0}^{i_1} g_{i+\frac{1}{2}}(c_i - c_k)^2 = \sum_{i=i_0}^{k-1} g_{i+\frac{1}{2}}(c_i - c_k)^2 + \sum_{i=k+1}^{i_1} g_{i+\frac{1}{2}}(c_i - c_k)^2. \qquad (5.21)
$$

We first note that

$$\begin{aligned}
&\sum_{i=k+1}^{i_1} g_{i+\frac{1}{2}}(c_i - c_k)^2 \\
\leq\ & C \sum_{i=k+1}^{i_1} \sum_{j=k}^{i-1} g_{i+\frac{1}{2}}(c_{j+1} - c_j)^2 \\
=\ & C \sum_{i=k+1}^{i_1} \sum_{j=k}^{i-1} \left(1 + \frac{g_{i+\frac{1}{2}} - g_{j+\frac{1}{2}}}{g_{j+\frac{1}{2}}}\right) g_{j+\frac{1}{2}}(c_{j+1} - c_j)^2. \qquad (5.22)
\end{aligned}$$

But from the definition of $g(x)$ in (5.8), we have

$$(1 + \frac{g_{i+\frac{1}{2}} - g_{j+\frac{1}{2}}}{g_{j+\frac{1}{2}}}) \leq e^{-\alpha(i-j)h} \leq e^{\alpha(i_1-i_0)h} \leq C, \qquad (5.23)$$

and using this in (5.22), we get

$$\begin{aligned}
\sum_{i=k+1}^{i_1} g_{i+\frac{1}{2}}(c_i - c_k)^2 \quad &\leq \quad C \sum_{i=k+1}^{i_1} \sum_{j=k}^{i-1} g_{j+\frac{1}{2}}(c_{j+1} - c_j)^2 \\
&\leq \quad C \sum_{j=k}^{i_1-1} g_{j+\frac{1}{2}}(c_{j+1} - c_j)^2, \qquad (5.24)
\end{aligned}$$

where C depends on $(i_1 - i_0)$.

Using similar arguments we can show that

$$\sum_{i=i_0}^{k-1} g_{i+\frac{1}{2}}(c_k - c_i)^2 \quad \leq \quad C(k - 1 - i_0) \sum_{j=i_0}^{k-1} g_{j+\frac{1}{2}}(c_{j+1} - c_j)^2, \qquad (5.25)$$

where C depends on $(i_1 - i_0)$. Therefore, combining (5.21), (5.24), and (5.25) we have

$$\sum_{i=i_0}^{i_1} g_{i+\frac{1}{2}}(c_i - c_k)^2 \leq C \sum_{i=i_0}^{i_1} g_{j+\frac{1}{2}}(c_{j+1} - c_j)^2, \qquad (5.26)$$

where C depends on $(i_i - i_0)$. Using similar arguments, we can prove (5.26) for all integers i_0, i_1 such that $i_0 < i_1$. $\square$

Lemma 5.4 *Suppose* $v(x) = \sum_{i\in\mathbb{Z}} c_i^h \phi_i^h(x)$. *Then*

(a) there are positive constants C_1, C_2, *independent of* v, h *and* j, *but may depend on* κ, *such that*

$$C_1 h \sum_{i\in A_j^h} g_i(c_i^h)^2 \leq \int_{I_j^h} g v^2\, dx \leq C_2 h \sum_{i\in A_j^h} g_i(c_i^h)^2; \qquad (5.27)$$

(b) there is a positive constant C, independent of v and h, such that

$$\frac{1}{h}\sum_{i\in\mathbb{Z}^n} g_{i+\frac{1}{2}}(c_{i+1}^h - c_i^h)^2 \le C\int_{\mathbb{R}} g{v'}^2\,dx. \tag{5.28}$$

Proof. (a) Consider $j \in \mathbb{Z}$ and the corresponding A_j^h such that, for $i \in A_j^h$, $H < x_i^h$. Let $g_M = \max_{i\in A_j^h} g(x_i^h)$ and $g_m = \min_{i\in A_j^h} g(x_i^h)$. Then, it is easy to check that $\frac{g_M}{g_m} \le C$, where C depends κ. Now, using (5.19), we have

$$\begin{aligned} h\sum_{i\in A_j^h} g_i(c_i^h)^2 &\le g_M h\sum_{i\in A_j^h}(c_i^h)^2 \\ &\le \frac{g_M}{C_1 g_m}\int_{I_j^h} gv^2\,dx \\ &\le C\int_{I_j^h} gv^2\,dx. \end{aligned} \tag{5.29}$$

Using a similar argument, we get

$$\int_{I_j^h} gv^2\,dx \le C\sum_{i\in A_j^h} g_i(c_i^h)^2.$$

Combining the above with (5.29) gives the required result. Using similar arguments, we can prove (5.27) for any $j \in \mathbb{Z}$. □

(b) Let $u = \sum_{i\in\mathbb{Z}} c_i\phi_i(x)$. Then from (5.14) and (5.15) with $h = 1$, we have

$$c_i = \Psi_i^1(u) = \int_{i-1}^{i+1} \psi_i^1(x)u(x)\,dx, \tag{5.30}$$

and therefore,

$$\begin{aligned} c_{i+1} - c_i &= \int_i^{i+2} \psi_{i+1}^1(x)u(x)\,dx - \int_{i-1}^{i+1} \psi_i^1(x)u(x)\,dx \\ &= \int_{i-1}^{i+2} (\psi_{i+1}^1(x) - \psi_i^1(x))u(x)\,dx. \end{aligned} \tag{5.31}$$

Let $F(x) = \int_{i-1}^{x}[\psi_{i+1}^1(t) - \psi_i^1(t)]\,dt$. Using translation and (5.12), it is easily seen that

$$\int_{i-1}^{i+1} \psi_i^1(t)\,dt = \int_i^{i+2} \psi_{i+1}^1(t)\,dt = 1,$$

and therefore, $F(i-1) = F(i+2) = 0$. Also, using the Schwartz inequality and (5.11), we can show that

$$\int_{i-1}^{i+2} F^2\,dx \le C.$$

Now, using the above bound, integrating (5.31) by parts, and using the Schwartz inequality, we get

$$(c_{i+1} - c_i)^2 = (\int_{i-1}^{i+2} Fu'\,dx)^2 \leq C \int_{i-1}^{i+2} {u'}^2\,dx. \tag{5.32}$$

Let $v = \sum_{i\in\mathbb{Z}} c_i^h \phi_i^h(x)$. Then by a standard scaling argument, we have

$$\int_{x_{i-1}^h}^{x_{i+2}^h} (v'(x))^2\,dx = \frac{1}{h}\int_{i-1}^{i+1} (u'(y))^2\,dy, \tag{5.33}$$

where $u(y) = \sum_{i\in\mathbb{Z}} c_i^h \phi_i(y)$. Therefore, from (5.32) and (5.33) we have

$$\frac{1}{h}(c_{i+1}^h - c_i^h)^2 \leq C \int_{x_{i-1}^h}^{x_{i+2}^h} {v'}^2\,dx. \tag{5.34}$$

From the definition of $g(x)$, we can show that for $x \in (x_{i-1}^h, x_{i+2}^h)$, $(1 + \frac{g_{i+1/2} - g(x)}{g(x)}) \leq C$. Therefore,

$$\begin{aligned}
\frac{1}{h} g_{i+\frac{1}{2}} (c_{i+1}^h - c_i^h)^2 &\leq C \int_{x_{i-1}^h}^{x_{i+2}^h} g{v'}^2 (1 + \frac{g_{i+\frac{1}{2}} - g}{g})\,dx \\
&\leq C \int_{x_{i-1}^h}^{x_{i+2}^h} g{v'}^2\,dx,
\end{aligned}$$

and hence,

$$\frac{1}{h}\sum_{i\in\mathbb{Z}} g_{i+\frac{1}{2}} (c_{i+1}^h - c_i^h)^2 \leq C \sum_{i\in\mathbb{Z}} \int_{x_{i-1}^h}^{x_{i+2}^h} g{v'}^2\,dx \leq C \int_{\mathbb{R}} g{v'}^2\,dx,$$

which is the required result. □

Remark 5.6 We note that it is possible to show that

$$\int_{\mathbb{R}} g{v'}^2\,dx \leq C \frac{1}{h} \sum_{i\in\mathbb{Z}^n} g_{i+\frac{1}{2}} (c_{i+1}^h - c_i^h)^2,$$

and together with (5.28) we see that $\frac{1}{h}\sum_{i\in\mathbb{Z}^n} g_{i+\frac{1}{2}}(c_{i+1}^h - c_i^h)^2$ is equivalent to $|v|_{H^1(\mathbb{R})}^2$. The proof of this fact is easier than the proof of (5.28), and we do got provide the proof here.

A perturbed bilinear form $B_\Theta(u, v)$ and related results:

For a given $\Theta \geq 1$, we now consider the bilinear form

$$B_\Theta^{\mathbb{R}}(u, v) \equiv B^{\mathbb{R}}(u, v) + \Theta D^{\mathbb{R}}(u, v),$$

where

$$B^{\mathbb{R}}(u,v) = \int_{\mathbb{R}} (u'v' + uv)\,dx \quad \text{and} \quad D^{\mathbb{R}}(u,v) = \int_{\mathbb{R}} uv\,dx.$$

We will write $B_\Theta(u,v) \equiv B_\Theta^{\mathbb{R}}(u,v)$, but will use $B_\Theta^F(u,v)$ when the domain of integration is F instead of $\mathbb{R}$. Also we will use $D^F(u,v)$, where $\mathbb{R}$ is replaced by a domain F in the definition of $D^{\mathbb{R}}(u,v)$.

Let $H^1_{g,\Theta}$ and $H^1_{g^{-1},\Theta}$ be Hilbert spaces defined as

$$\begin{aligned} H^1_{g,\Theta} &= \{u : \|u\|^2_{1,g,\Theta} \equiv \int_{\mathbb{R}} g u'^2\,dx + (1+\Theta)\int_{\mathbb{R}} g u^2\,dx < \infty\}; \\ H^1_{g^{-1},\Theta} &= \{u : \|u\|^2_{1,g^{-1},\Theta} \equiv \int_{\mathbb{R}} g^{-1} u'^2\,dx + (1+\Theta)\int_{\mathbb{R}} g^{-1} u^2\,dx < \infty\}. \end{aligned}$$

We will choose Θ later. The choice of Θ, along with the choices of γ and α, mentioned before, is important for the main result of this section. We assume that $\frac{\alpha^2}{\bar{\Theta}} < 1$ where $\bar{\Theta} = 1 + \Theta$.

We will often suppress Θ in $\|u\|_{1,g,\Theta}$ and $\|u\|_{1,g^{-1},\Theta}$ and instead write $\|u\|_{1,g}$ and $\|u\|_{1,g^{-1}}$ respectively. We will also use the fact that $|g'/g| \le \alpha$, which is obvious from the definition of $g(x)$.

Remark 5.7 The space $H^1_{g,\Theta}$ is directed towards obtaining interior estimates of e'_h, *i.e.*, e'_h is locally characterized through the use of the space $H_{g,\Theta}$.

We now consider $B_\Theta(\cdot,\cdot) : H^1_{g,\Theta} \times H^1_{g^{-1},\Theta} \to \mathbb{R}$.

Lemma 5.5 *The bilinear form $B_\Theta(\cdot,\cdot)$ is bounded on $H^1_{g,\Theta} \times H^1_{g^{-1},\Theta}$, i.e.,*

$$B_\Theta(u,v) \le C\|u\|_{1,g}\|v\|_{1,g^{-1}}, \quad \textit{for all } u \in H^1_{g,\Theta},\ v \in H^1_{g^{-1},\Theta}.$$

Proof. Let $u \in H^1_{g,\Theta}$ and $v \in H^1_{g^{-1},\Theta}$. Then

$$\begin{aligned} B_\Theta(u,v) &= \int_{\mathbb{R}} [u'v' + (1+\Theta)uv]\,dx \\ &= \int_{\mathbb{R}} [g^{1/2}u'g^{-1/2}v' + (1+\Theta)^{1/2}g^{1/2}u(1+\Theta)^{1/2}g^{-1/2}v]\,dx \\ &\le C\Big[\int_{\mathbb{R}} (gu'^2 + (1+\Theta)gu^2)\,dx\Big]^{1/2}\Big[\int_{\mathbb{R}} (g^{-1}v'^2 + (1+\Theta)g^{-1}v^2)\,dx\Big]^{1/2} \\ &= C\|u\|_{1,g}\|v\|_{1,g^{-1}}. \quad \square \end{aligned}$$

Lemma 5.6 *Suppose $\frac{\alpha^2}{\bar{\Theta}} < 1$. Then there is a constant $C > 0$, which depends on $\frac{\alpha^2}{\bar{\Theta}}$, such that*

$$\inf_{u \in H^1_{g,\Theta}} \sup_{v \in H^1_{g^{-1},\Theta}} \frac{B_\Theta(u,v)}{\|u\|_{1,g}\|v\|_{1,g^{-1}}} \ge C > 0.$$

Proof. Suppose $u \in H^1_{g,\Theta}$. We consider $v = gu$. Now,

$$\begin{aligned} B_\Theta(u,v) &= \int_{\mathbb{R}} [u'v' + \bar{\Theta}uv]\,dx \\ &= \int_{\mathbb{R}} [u'(gu' + g'u) + \bar{\Theta}gu^2]\,dx \\ &= \int_{\mathbb{R}} [gu'^2 + \bar{\Theta}gu^2]\,dx + \int_{\mathbb{R}} uu'g'\,dx. \qquad (5.35) \end{aligned}$$

Now, for $\epsilon > 0$,

$$\begin{aligned} |\int_{\mathbb{R}} uu'g'\,dx| &= |\int_{\mathbb{R}} guu'(\frac{g'}{g})\,dx| \\ &\leq \alpha \int_{\mathbb{R}} |g^{1/2}ug^{1/2}u'|\,dx \\ &\leq \alpha[\epsilon \int_{\mathbb{R}} gu'^2\,dx + \frac{1}{\epsilon}\int_{\mathbb{R}} gu^2\,dx], \end{aligned}$$

and, therefore from (5.35), we get

$$\begin{aligned} B_\Theta(u,v) &\geq \int_{\mathbb{R}} (gu'^2 + \bar{\Theta}gu^2)\,dx - \alpha[\epsilon\int_{\mathbb{R}} gu'^2\,dx + \frac{1}{\epsilon}\int_{\mathbb{R}} gu^2\,dx] \\ &= (1-\alpha\epsilon)\int_{\mathbb{R}} gu'^2\,dx + (1 - \frac{\alpha}{\epsilon\bar{\Theta}})\int_{\mathbb{R}} \bar{\Theta}gu^2\,dx. \qquad (5.36) \end{aligned}$$

We choose ϵ such that $\alpha\epsilon < 1$ and $\frac{\alpha}{\epsilon\bar{\Theta}} < 1$, and therefore from (5.36), we have

$$B_\Theta(u,v) \geq C_1\|u\|^2_{1,g}, \qquad (5.37)$$

where

$$C_1 = \min[(1-\alpha\epsilon), (1 - \frac{\alpha}{\epsilon\bar{\Theta}})] > 0. \qquad (5.38)$$

We next show that $\|v\|_{1,g^{-1}} \leq C_2\|u\|_{1,g}$. First note that

$$\begin{aligned} \int_{\mathbb{R}} g^{-1}v'^2\,dx &= \int_{\mathbb{R}} g^{-1}(gu' + g'u)^2\,dx \\ &= \int_{\mathbb{R}} gu'^2\,dx + \int_{\mathbb{R}} g^{-1}g'^2u^2\,dx + 2\int_{\mathbb{R}} g'uu'\,dx. \qquad (5.39) \end{aligned}$$

Now,

$$\int_{\mathbb{R}} g^{-1}g'^2u^2\,dx = \int_{\mathbb{R}} g(\frac{g'}{g})^2u^2\,dx \leq \alpha^2\int_{\mathbb{R}} gu^2\,dx, \qquad (5.40)$$

and

$$\begin{aligned} 2\int_{\mathbb{R}} g'uu'\,dx = 2\int_{\mathbb{R}} g(\frac{g'}{g})uu'\,dx &\leq 2\int_{\mathbb{R}} |\alpha g^{1/2}ug^{1/2}u'|\,dx \\ &\leq \int_{\mathbb{R}} (gu'^2 + \alpha^2gu^2)\,dx. \qquad (5.41) \end{aligned}$$

Therefore using (5.40) and (5.41) in (5.39), we get

$$\int_{\mathbb{R}} g^{-1}{v'}^2\,dx \le 2\int_{\mathbb{R}} g{u'}^2\,dx + \frac{2\alpha^2}{\bar{\Theta}}\int_{\mathbb{R}} \bar{\Theta} g u^2\,dx. \tag{5.42}$$

Thus, combining

$$\bar{\Theta}\int_{\mathbb{R}} g^{-1}v^2\,dx = \bar{\Theta}\int_{\mathbb{R}} g^{-1}g^2u^2\,dx = \bar{\Theta}\int_{\mathbb{R}} gu^2\,dx$$

with (5.42), we get

$$\|v\|^2_{1,g^{-1}} \quad \le \quad 2\int_{\mathbb{R}} g{u'}^2\,dx + (1+\frac{2\alpha^2}{\bar{\Theta}})\int_{\mathbb{R}} \bar{\Theta} g u^2\,dx \tag{5.43}$$

Since $\frac{\alpha^2}{\bar{\Theta}} < 1$, therefore, from (5.43) we have

$$\|v\|^2_{1,g^{-1}} \le 3\|u\|^2_{1,g}. \tag{5.44}$$

Thus, $v \in H^1_{g^{-1},\Theta}$, and combining (5.37) and (5.44) we get

$$\inf_{u\in H^1_{g,\Theta}}\sup_{v\in H^1_{g^{-1},\Theta}} \frac{B_\Theta(u,v)}{\|u\|_{1,g}\|v\|_{1,g^{-1}}} \ge C > 0$$

where

$$C = \frac{\min[(1-\alpha\epsilon),(1-\frac{\alpha}{\epsilon\bar{\Theta}})]}{\sqrt{3}}. \quad \square$$

We now prove the inf-sup condition on $S_h \times S_h$. In the proof, we will use the function $d_i(x), x \in I^h_k$ and $i \in A^h_k$ to denote the following similar functions:

$$\frac{g_i - g(x)}{\sqrt{g_i\, g(x)}},\ \frac{g_{i+\frac{1}{2}} - g(x)}{\sqrt{g_{i+\frac{1}{2}}\, g(x)}},\ \frac{g_{i+\frac{1}{2}} - g_{l_k+\frac{1}{2}}}{\sqrt{g_{l_k+\frac{1}{2}}\, g(x)}}$$

where $l_k \in A^h_k$. It is easily seen from the definition of $g(x)$ that

$$|d_i(x)| \le C\alpha h \tag{5.45}$$

Lemma 5.7 *Suppose $\frac{\alpha^2}{\bar{\Theta}} < C_1$ and $\alpha h < C_2$, where C_1, C_2 are sufficiently small. Then there is a constant $C > 0$, independent of u, v, and h, but may depend on κ and $\frac{\alpha^2}{\bar{\Theta}}$, such that for h small enough,*

$$\inf_{u\in S_h}\sup_{v\in S_h} \frac{B_\Theta(u,v)}{\|u\|_{1,g}\|v\|_{1,g^{-1}}} \ge C > 0. \tag{5.46}$$

Proof. Let $u = \sum_{i\in\mathbb{Z}} c_i^h \phi_i^h$ in S_h such that $\|u\|_{1,g} \leq \infty$. Then for $x \in I_k^h$, we have $u = \sum_{i\in A_k^h} c_i^h \phi_i^h$. Since $\sum_{i\in A_k^h} {\phi_i^h}'(x) = 0$ for $x \in I_k^h$, we have

$$u'(x) = \sum_{i\in A_k^h} c_i^h {\phi_i^h}'(x) = \sum_{i\in A_k^h} (c_i^h - c_{l_k}^h){\phi_i^h}'(x), \quad x \in I_k^h,$$

where $l_k \in A_k^h$ is a fixed integer for given k.

We now choose $v = \sum_{i\in\mathbb{Z}} c_i^h g_{i+\frac{1}{2}} \phi_i^h$ in S_h, and as before, for $x \in I_k^h$,

$$\begin{aligned} v'(x) &= \sum_{i\in A_k^h} c_i^h g_{i+\frac{1}{2}} {\phi_i^h}'(x) \\ &= \sum_{i\in A_k^h} (c_i^h g_{i+\frac{1}{2}} - c_{l_k}^h g_{l_k+\frac{1}{2}}){\phi_i^h}'(x) \\ &= \sum_{i\in A_k^h} (c_i^h - c_{l_k}^h) g_{i+\frac{1}{2}} {\phi_i^h}'(x) + c_{l_k}^h \sum_{i\in A_k^h} (g_{i+\frac{1}{2}} - g_{l_k+\frac{1}{2}}){\phi_i^h}'(x). \end{aligned}$$

Now,

$$\int_{\mathbb{R}} u'v'\,dx = \int_{\mathbb{R}} g{u'}^2\,dx + \int_{\mathbb{R}} u'(v' - gu')\,dx. \tag{5.47}$$

For $\epsilon > 0$, we have

$$\begin{aligned} \left|\int_{\mathbb{R}} u'(v' - gu')\,dx\right| &= \left|\int_{\mathbb{R}} g^{1/2} u' \frac{(v' - gu')}{g^{1/2}}\,dx\right| \\ &\leq \epsilon \int_{\mathbb{R}} g{u'}^2\,dx + \frac{1}{\epsilon}\int_{\mathbb{R}} \frac{(v' - gu')^2}{g}\,dx. \end{aligned} \tag{5.48}$$

Now, from the definition of v' and u',

$$\begin{aligned} \int_{I_k^h} \frac{1}{g}(v' - gu')^2\,dx &= \int_{I_k^h} [\sum_{i\in A_k^h} (c_i^h - c_{l_k}^h) \frac{g_{i+\frac{1}{2}} - g}{g^{1/2}} {\phi_i^h}' \\ &\quad + c_{l_k}^h \sum_{i\in A_k^h} \frac{g_{i+\frac{1}{2}} - g_{l_k+\frac{1}{2}}}{g^{1/2}} {\phi_i^h}']^2\,dx \\ &\leq C\int_{I_k^h} [\sum_{i\in A_k^h} (c_i^h - c_{l_k}^h) \frac{g_{i+\frac{1}{2}} - g}{g^{1/2}} {\phi_i^h}']^2\,dx \\ &\quad + C\int_{I_k^h} (c_{l_k}^h)^2 \sum_{i\in A_k^h} \frac{g_{i+\frac{1}{2}} - g_{l_k+\frac{1}{2}}}{g^{1/2}} {\phi_i^h}']^2\,dx. \end{aligned} \tag{5.49}$$

The first term of the RHS of the above inequality, employing (5.45) and (5.20), gives

$$
\begin{aligned}
& \int_{I_k^h}[\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)\frac{g_{i+\frac{1}{2}} - g}{g^{1/2}}\phi_i^{h\prime}]^2\,dx \\
= \quad & \int_{I_k^h}[\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)(g_{i+\frac{1}{2}})^{1/2}\frac{g_{i+\frac{1}{2}} - g}{(g_{i+\frac{1}{2}})^{1/2}g^{1/2}}\phi_i^{h\prime}]^2\,dx \\
\leq \quad & C\int_{I_k^h}\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)^2 g_{i+\frac{1}{2}} d_i^2(\phi_i^{h\prime})^2\,dx \\
\leq \quad & C\alpha^2 h^2\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)^2 g_{i+\frac{1}{2}}\int_{I_k^h}(\phi_i^{h\prime})^2\,dx \\
\leq \quad & C\alpha^2 h^2\frac{1}{h}\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)^2 g_{i+\frac{1}{2}} \\
\leq \quad & C\alpha^2 h^2\frac{1}{h}\sum_{i,(i+1)\in A_k^h}(c_{i+1}^h - c_i^h)^2 g_{i+\frac{1}{2}}, \qquad (5.50)
\end{aligned}
$$

where C is independent of $\alpha,\ h$, but depends on κ.

The second term of the RHS of (5.49), employing (5.45), gives

$$
\begin{aligned}
& (c_{l_k}^h)^2\int_{I_k^h}[\sum_{i\in A_k^h}\frac{g_{i+\frac{1}{2}} - g_{l_k+\frac{1}{2}}}{g^{1/2}}\phi_i^{h\prime}]^2\,dx \\
= \quad & (c_{l_k}^h)^2 g_{l_k+\frac{1}{2}}\int_{I_k^h}[\sum_{i\in A_k^h}\frac{g_{i+\frac{1}{2}} - g_{l_k+\frac{1}{2}}}{(g_{l_k+\frac{1}{2}})^{1/2}g^{1/2}}\phi_i^{h\prime}]^2\,dx \\
\leq \quad & C(c_{l_k}^h)^2 g_{l_k+\frac{1}{2}}\sum_{i\in A_k^h}\int_{I_k^h}d_i^2(\phi_i^{h\prime})^2\,dx \\
\leq \quad & C\alpha^2 h(c_{l_k}^h)^2 g_{l_k}, \qquad (5.51)
\end{aligned}
$$

where C depends on κ, but is independent of $\alpha,\ h$. Therefore, from (5.49), (5.50), and (5.51) we have

$$
\begin{aligned}
\int_{I_k^h}\frac{1}{g}(v' - gu')^2\,dx \quad \leq \quad & C\alpha^2 h^2\frac{1}{h}\sum_{i,(i+1)\in A_k^h}(c_{i+1}^h - c_i^h)^2 g_{i+\frac{1}{2}} \\
& + C\alpha^2 h(c_{l_k}^h)^2 g_{l_k}.
\end{aligned}
$$

Now summing the above inequality over $k\in\mathbb{Z}$, and using (5.27) and (5.28), we get

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{g}(v' - gu')^2 \, dx &= \sum_{k\in\mathbb{Z}} \int_{I_k^h} \frac{1}{g}(v' - gu')^2 \, dx \\
&\leq C\alpha^2 h^2 \frac{1}{h} \sum_{k\in\mathbb{Z}} \sum_{i,(i+1)\in A_k^h} (c_{i+1}^h - c_i^h)^2 g_{i+\frac{1}{2}} \\
&\quad + C\alpha^2 h \sum_{k\in\mathbb{Z}} \sum_{i\in A_k^h} (c_i^h)^2 g_i \\
&\leq C\alpha^2 h^2 \frac{1}{h} \sum_{i\in\mathbb{Z}} (c_{i+1}^h - c_i^h)^2 g_{i+\frac{1}{2}} \\
&\quad + C\alpha^2 \sum_{k\in\mathbb{Z}} \int_{I_k^h} gu^2 \, dx \\
&\leq C\alpha^2 h^2 \int_{\mathbb{R}} gu'^2 \, dx + C\alpha^2 \int_{\mathbb{R}} gu^2 \, dx. \qquad (5.52)
\end{aligned}$$

Then from (5.48) and (5.52) we have

$$\begin{aligned}
\int_{\mathbb{R}} u'(v' - gu') \, dx &\leq \epsilon \int_{\mathbb{R}} gu'^2 \, dx \\
&\quad + \frac{1}{\epsilon}[C\alpha^2 h^2 \int_{\mathbb{R}} gu'^2 \, dx + C\alpha^2 \int_{\mathbb{R}} gu^2 \, dx] \\
&= (\epsilon + \frac{C\alpha^2 h^2}{\epsilon}) \int_{\mathbb{R}} gu'^2 \, dx + \frac{C\alpha^2}{\epsilon\bar{\Theta}} \int_{\mathbb{R}} \bar{\Theta} gu^2 \, dx. \qquad (5.53)
\end{aligned}$$

We next consider

$$\bar{\Theta} \int_{\mathbb{R}} uv \, dx = \bar{\Theta} \int_{\mathbb{R}} gu^2 \, dx + \bar{\Theta} \int_{\mathbb{R}} u(v - gu) \, dx. \qquad (5.54)$$

For $\epsilon_1 > 0$, we have

$$\begin{aligned}
\left| \int_{\mathbb{R}} u(v - gu) \, dx \right| &= \left| \int_{\mathbb{R}} g^{1/2} u \frac{v - gu}{g^{1/2}} \, dx \right| \\
&\leq \epsilon_1 \int_{\mathbb{R}} gu^2 \, dx + \frac{1}{\epsilon_1} \int_{\mathbb{R}} \frac{(v - gu)^2}{g} \, dx. \qquad (5.55)
\end{aligned}$$

Now,

$$
\begin{aligned}
\int_{I_k^h} \frac{(v-gu)^2}{g}\,dx &= \int_{I_k^h} \frac{1}{g}[\sum_{i\in A_k^h} c_i^h (g_i - g)\phi_i^h]^2\,dx \\
&= \int_{I_k^h} [\sum_{i\in A_k^h} c_i^h {g_i}^{1/2} \frac{(g_i-g)}{{g_i}^{1/2} g^{1/2}} \phi_i^h]^2\,dx \\
&\leq C \int_{I_k^h} \sum_{i\in A_k^h} (c_i^h)^2 g_i d_i^2 {\phi_i^h}^2\,dx \\
&\leq C\alpha^2 h^2 h \sum_{i\in A_k^h} (c_i^h)^2 g_i.
\end{aligned}
$$

Therefore using (5.27), we get

$$
\begin{aligned}
\int_{\mathbb{R}} \frac{(v-gu)^2}{g}\,dx &= \sum_{k\in\mathbb{Z}} \int_{I_k^h} \frac{(v-gu)^2}{g}\,dx \\
&\leq \sum_{k\in\mathbb{Z}} C\alpha^2 h^2 h \sum_{i\in A_k^h} (c_i^h)^2 g_i \\
&\leq C\alpha^2 h^2 \int_{\mathbb{R}} gu^2\,dx. \qquad (5.56)
\end{aligned}
$$

Thus from (5.55), (5.56), we have

$$
\bar{\Theta}|\int_{\mathbb{R}} u(v-gu)\,dx| \leq (\epsilon_1 + \frac{C\alpha^2h^2}{\epsilon_1}) \int_{\mathbb{R}} \bar{\Theta} gu^2\,dx, \qquad (5.57)
$$

and combining (5.47), (5.53), (5.54), and (5.57), we get

$$
\begin{aligned}
|B_\Theta(u,v)| &\geq \int_{\mathbb{R}} g{u'}^2\,dx + \bar{\Theta}\int_{\mathbb{R}} gu^2\,dx \\
&\quad -|\int_{\mathbb{R}} u'(v'-gu')\,dx| - \bar{\Theta}|\int_{\mathbb{R}} u(v-gu)\,dx| \\
&\geq (1-\epsilon-\frac{C\alpha^2h^2}{\epsilon}) \int_{\mathbb{R}} g{u'}^2\,dx \\
&\quad +(1-\epsilon_1-\frac{C\alpha^2h^2}{\epsilon_1}-\frac{C\alpha^2}{\epsilon\bar{\Theta}}) \int_{\mathbb{R}} \bar{\Theta} gu^2\,dx.
\end{aligned}
$$

Now we can choose ϵ and ϵ_1, for sufficiently small h, such that

$$
|B_\Theta(u,v)| \geq C_1 \|u\|_{1,g}^2, \qquad (5.58)
$$

where $C_1 > 0$, since $\frac{\alpha^2}{\bar{\Theta}} << 1$, $\alpha h << 1$ by assumption.

We now show that $\|v\|_{1,g^{-1}} \leq C\|u\|_{1,g}$. From the definition of v', we have

$$\begin{aligned}
\int_{I_k^h} g^{-1}{v'}^2\,dx &= \int_{I_k^h} g^{-1}[\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)g_{i+\frac{1}{2}}{\phi_i^h}' \\
&\quad + c_{l_k}\sum_{i\in A_k^h}(g_{i+\frac{1}{2}} - g_{l_k+\frac{1}{2}}){\phi_i^h}']^2\,dx \\
&\leq \int_{I_k^h}[\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)\frac{g_{i+\frac{1}{2}}}{g^{1/2}}{\phi_i^h}']^2\,dx \\
&\quad + c_{l_k}^2\int_{I_k^h}[\sum_{i\in A_k^h}\frac{(g_{i+\frac{1}{2}} - g_{l_k+\frac{1}{2}})}{g^{1/2}}{\phi_i^h}']^2\,dx. \qquad (5.59)
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_{I_k^h}[\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)\frac{g_{i+\frac{1}{2}}}{g^{1/2}}{\phi_i^h}']^2\,dx \\
= &\int_{I_k^h}[\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)g^{1/2}{\phi_i^h}' + \sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)g_{i+\frac{1}{2}}^{1/2}\left(\frac{g_{i+\frac{1}{2}} - g}{g_{i+\frac{1}{2}}^{1/2}g^{1/2}}\right){\phi_i^h}']^2\,dx \\
\leq\ & C\int_{I_k^h} g[\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h){\phi_i^h}']^2\,dx + C\int_{I_k^h}\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)^2 g_{i+\frac{1}{2}}d_i^2({\phi_i^h}')^2\,dx \\
\leq\ & C\int_{I_k^h} g{u'}^2\,dx + C\int_{I_k^h}\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)^2 g_{i+\frac{1}{2}}d_i^2({\phi_i^h}')^2\,dx. \qquad (5.60)
\end{aligned}$$

Also using (5.45) and (5.20), we have

$$\begin{aligned}
\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)^2 g_{i+\frac{1}{2}}\int_{I_k^h} d_i^2({\phi_i^h}')^2\,dx &\leq C\alpha^2 h^2\frac{1}{h}\sum_{i\in A_k^h}(c_i^h - c_{l_k}^h)^2 g_{i+\frac{1}{2}} \\
&\leq C\alpha^2 h^2\frac{1}{h}\sum_{i,i+1\in A_k^h}(c_{i+1}^h - c_i^h)^2 g_{i+\frac{1}{2}}. \quad (5.61)
\end{aligned}$$

Therefore, using (5.60), (5.61) and (5.51) in (5.59), we get

$$\begin{aligned}
&\int_{I_k^h} g^{-1}{v'}^2\,dx \\
\leq\ & C\int_{I_k^h} g{u'}^2\,dx + C\alpha^2 h(c_{l_k}^h)^2 g_{l_k} + C\alpha^2 h^2\frac{1}{h}\sum_{i,i+1\in A_k^h}(c_{i+1}^h - c_i^h)^2 g_{i+\frac{1}{2}} \\
\leq\ & C\int_{I_k^h} g{u'}^2\,dx + C\alpha^2 h\sum_{i\in A_k^h}(c_i^h)^2 g_i + C\alpha^2 h^2\frac{1}{h}\sum_{i,i+1\in A_k^h}(c_{i+1}^h - c_i^h)^2 g_{i+\frac{1}{2}} \\
\leq\ & C\int_{I_k^h} g{u'}^2\,dx + C\alpha^2\int_{I_k^h} gu^2\,dx + C\alpha^2 h^2\frac{1}{h}\sum_{i,i+1\in A_k^h}(c_{i+1}^h - c_i^h)^2 g_{i+\frac{1}{2}}.
\end{aligned}$$

Now, summing the above inequality for all k and using (5.20), we get

$$\int_{\mathbb{R}} g^{-1} {v'}^2 \, dx \quad \leq \quad C(1+\alpha^2 h^2) \int_{\mathbb{R}} g {u'}^2 \, dx + C\alpha^2 \int_{\mathbb{R}} g u^2 \, dx. \qquad (5.62)$$

Again,

$$\int_{\mathbb{R}} g^{-1} v^2 \, dx \quad = \quad \int_{\mathbb{R}} g^{-1} (\sum_{i\in\mathbb{Z}} c_i^h g_i \phi_i^h)^2 \, dx = \int_{\mathbb{R}} (\sum_{i\in\mathbb{Z}} c_i^h \frac{g_i}{g^{1/2}} \phi_i^h)^2 \, dx. \quad (5.63)$$

Now using (5.45), we get

$$\begin{aligned}
& \int_{I_k^h} (\sum_{i\in A_k^h} c_i^h \frac{g_i}{g^{1/2}} \phi_i^h)^2 \, dx \\
= \quad & \int_{I_k^h} [\sum_{i\in A_k^h} c_i^h g^{1/2} \phi_i^h + \sum_{i\in A_k^h} c_i^h \frac{g_i - g}{g^{1/2}} \phi_i^h]^2 \, dx \\
\leq \quad & C \int_{I_k^h} (\sum_{i\in A_k^h} c_i^h g^{1/2} \phi_i^h)^2 \, dx + C \int_{I_k^h} \sum_{i\in A_k^h} (c_i^h)^2 g_i \left(\frac{g_i - g}{g^{1/2} g_i^{1/2}} \right)^2 {\phi_i^h}^2 \, dx \\
\leq \quad & C \int_{I_k^h} g u^2 \, dx + C \sum_{i\in A_k^h} (c_i^h)^2 g_i \int_{I_k^h} d_i^2 {\phi_i^h}^2 \, dx \\
\leq \quad & C \int_{I_k^h} g u^2 \, dx + C\alpha^2 h^2 h \sum_{i\in A_k^h} (c_i^h)^2 g_i \\
\leq \quad & C \int_{I_k^h} g u^2 \, dx + C\alpha^2 h^2 C \int_{I_k^h} g u^2 \, dx \\
\leq \quad & C(1+\alpha^2 h^2) \int_{I_k^h} g u^2 \, dx,
\end{aligned}$$

and therefore, from (5.63) and the above inequality,

$$\begin{aligned}
\int_{\mathbb{R}} g^{-1} v^2 \, dx \quad & \leq \quad C \sum_{k\in\mathbb{Z}} \int_{I_k^h} (\sum_{i\in A_k^h} c_i^h \frac{g_i}{g^{1/2}} \phi_i^h)^2 \, dx \\
& \leq \quad C(1+\alpha^2 h^2) \sum_{k\in\mathbb{Z}} \int_{I_k^h} g u^2 \, dx \\
& = \quad C(1+\alpha^2 h^2) \int_{\mathbb{R}} g u^2 \, dx.
\end{aligned}$$

Thus combining (5.62) and above inequality, we have

$$\begin{aligned}
\|v\|_{1,g^{-1}}^2 &= \int_{\mathbb{R}} g^{-1}v'^2\,dx + \bar{\Theta}\int_{\mathbb{R}} g^{-1}v^2\,dx \\
&\leq C(1+\alpha^2h^2)\int_{\mathbb{R}} gu'^2\,dx + C\alpha^2\int_{\mathbb{R}} gu^2\,dx \\
&\qquad +C(1+\alpha^2h^2)\int_{\mathbb{R}} \bar{\Theta}gu^2\,dx \\
&\leq C(1+\alpha^2h^2)\int_{\mathbb{R}} gu'^2\,dx \\
&\qquad +[\frac{C\alpha^2}{\bar{\Theta}} + C(1+\alpha^2h^2)]\int_{\mathbb{R}} \bar{\Theta}gu^2\,dx \\
&\leq C(1+\alpha^2h^2+\frac{\alpha^2}{\bar{\Theta}})\|u\|_{1,g}^2 \leq C_2\|u\|_{1,g}^2. \qquad (5.64)
\end{aligned}$$

Finally, combining (5.58) and (5.64) we get the desired result. □

Projection with respect to $B_\Theta(u,v)$*:*

Suppose $u \in H^1_{g,\Theta}$ and let $P_\Theta u$ be the projection of u onto S_h defined by

$$B_\Theta(P_\Theta u, v) = B_\Theta(u,v), \quad \text{for all } v \in S_h.$$

The projection $P_\Theta u$ exists (see [11]), and it is clear from Lemmas 5.7 and 5.5 that

$$\|P_\Theta u\|_{1,g} \leq C \sup_{v\in S_h} \frac{B_\Theta(u,v)}{\|v\|_{1,g^{-1}}} \leq C\|u\|_{1,g}. \qquad (5.65)$$

We first note that for fixed h, α, and Θ, the polynomials belong to the space $H^1_{g,\Theta}$. Moreover, for fixed h, α, and Θ, we can also show, using (5.27) and Remark 5.6, that $\tilde{\mathcal{I}}_h(x^{k+1}) \in H^1_{g,\Theta}$, where $\tilde{\mathcal{I}}_h(x^{k+1})$ is the interpolant of x^{k+1}, as defined in (4.35).

We now present some simple facts about polynomials and periodic functions.

Lemma 5.8 *Let the shape functions* $\{\phi_i^h\}_{i\in Z}$ *be reproducing of order* k*. Then*

$$\text{(a) } P_\Theta x^i = x^i, \quad 0 \leq i \leq k \qquad (5.66)$$

$$\text{(b) } P_\Theta \tilde{\mathcal{I}}_h(x^{k+1}) = \tilde{\mathcal{I}}_h(x^{k+1}), \qquad (5.67)$$

where $\tilde{\mathcal{I}}_h(x^{k+1})$ *is the interpolant of* x^{k+1} *as defined in Section 4.*

The proofs of these facts are immediate.

Lemma 5.9 *Suppose* $f \in H^1_{g,\Theta}$ *is periodic, i.e.,* $f(x + x_k^h) = f(x)$ *for all* k*. Then* $P_\Theta f$ *is also periodic.*

Proof. Let $\tilde{f}(x) = f(x + x_k^h)$. Then $[P_\Theta \tilde{f}](x) = [P_\Theta f](x + x_k^h)$. Now $f(x) = \tilde{f}(x)$ since f is periodic, and thus from the uniqueness of the the projection P_Θ, we have $[P_\Theta f](x + x_k^h) = [P_\Theta f](x)$, *i.e.,* $P_\Theta f$ is periodic. □

Remark 5.8 We note that if

$$v(x) = \sum_{i\in\mathbb{Z}} c_i^h \phi_i^h(x)$$

is a periodic function, *i.e.*, $v(x + x_k^h) = v(x)$ for any k, then v is a constant. This could be shown as follows: Since $v(x + x_k^h) = v(x)$, we have

$$v(x + x_k^h) = \sum_{i\in\mathbb{Z}} c_i^h \phi_i^h(x + x_k^h) = \sum_{i\in\mathbb{Z}} c_{i+k}^h \phi_i^h(x) = \sum_{i\in\mathbb{Z}} c_i^h \phi_i^h(x) = v(x),$$

which implies that

$$\sum_{i\in\mathbb{Z}} [c_{i+k}^h - c_i^h]\phi_i^h(x) = 0, \quad \text{for all } x \in \mathbb{R}.$$

Using (5.19), we can show that $\{\phi_i^h\}_{i\in\mathbb{Z}}$ are linearly independent in $\mathbb{R}$. Thus we infer from above that $c_{i+k}^h = c_i^h = C$ (constant), for all $i \in \mathbb{Z}$. Recalling that $\{\phi_i^h\}_{i\in\mathbb{Z}}$ form a partition of unity, we get $v(x) = C\sum_{i\in\mathbb{Z}} \phi_i^h(x) = C$.

We now define

$$\xi_{k+1}^\Theta(x) \equiv x^{k+1} - P_\Theta x^{k+1}. \tag{5.68}$$

$\xi_{k+1}^\Theta(x)$ and the next result will play a central role in the final result of this section.

Lemma 5.10 *Let $\xi_{k+1}^\Theta(x)$ be as defined in (5.68) and consider $\xi_{k+1}^h(x) = x^{k+1} - \sum_{i\in\mathbb{Z}}(x_i^h)^{k+1}\phi_i^h(x)$ as defined in (4.36). Then*

$${\xi_{k+1}^\Theta}'(x) = {\xi_{k+1}^h}'(x). \tag{5.69}$$

Proof. We first note, from the definition of $\tilde{\mathcal{I}}_h x^{k+1}$, that $\xi_{k+1}^h(x) = x^{k+1} - \tilde{\mathcal{I}}_h x^{k+1}$. Now, using (5.67), we have

$$\begin{aligned}
\xi_{k+1}^\Theta &= x^{k+1} - P_\Theta x^{k+1} \\
&= x^{k+1} - \tilde{\mathcal{I}}_h x^{k+1} + \tilde{\mathcal{I}}_h x^{k+1} - P_\Theta x^{k+1} \\
&= \xi_{k+1}^h - P_\Theta[x^{k+1} - \tilde{\mathcal{I}}_h x^{k+1}] \\
&= \xi_{k+1}^h - P_\Theta[\xi_{k+1}^h].
\end{aligned} \tag{5.70}$$

But we know from Lemma 4.1 that $\xi_{k+1}^h(x)$ is periodic, and therefore from Lemma 5.9 and Remark 5.8 we infer that $P_\Theta[\xi_{k+1}^h]$ is a constant. Thus, from (5.70), we get

$${\xi_{k+1}^\Theta}'(x) = {\xi_{k+1}^h}'(x),$$

which is the desired result. □

Proof of Theorem 5.1:
The proof will be given in several steps.

1. Let E be the extension operator satisfying (3.51). Then, for $x \in B_H \equiv B_H(0)$, we have

$$[u_0 - u_h](x) = [u_0 - P_\Theta(Eu_0) - \{Eu_h - P_\Theta(Eu_h)\} + \{P_\Theta(Eu_0) - P_\Theta(Eu_h)\}](x),$$

and therefore,

$$(u_0' - u_h{}')(x) = \{u_0 - P_\Theta(Eu_0)\}'(x) - \delta_h'(x) + \rho_h'(x), \tag{5.71}$$

where

$$\begin{aligned} \delta_h &= Eu_h - P_\Theta(Eu_h); && (5.72) \\ \rho_h &= P_\Theta(Eu_0) - P_\Theta(Eu_h). && (5.73) \end{aligned}$$

Since $u_0 = Eu_0$ in $B_H(0)$, from Taylor's Theorem we have

$$Eu_0(x) = \sum_{j=0}^{k} \frac{u_0^{(j)}(0)}{j!} x^j + \frac{u_0^{(k+1)}(0)}{(k+1)!} x^{k+1} + R_{k+1}(Eu_0)(x), \tag{5.74}$$

where $R_{k+1}(Eu_0)(x)$ is the remainder given by

$$R_{k+1}(Eu_0)(x) = \frac{1}{(k+1)!} \int_0^x (x-t)^{k+1} (Eu_0)^{(k+2)}(t)\, dt. \tag{5.75}$$

Since P_Θ is a linear operator, we have

$$P_\Theta(Eu_0)(x) = \sum_{j=0}^{k} \frac{u_0^{(j)}(0)}{j!} P_\Theta x^j + \frac{u_0^{(k+1)}(0)}{(k+1)!} P_\Theta x^{k+1} + P_\Theta R_{k+1}(Eu_0)(x). \tag{5.76}$$

We know from (5.66) that $P_\Theta x^j = x^j, 0 \le j \le k$. Therefore, by first subtracting (5.76) from (5.74), then differentiating the identity, and finally using (5.69), we have

$$\begin{aligned} & \{Eu_0 - P_\Theta(Eu_0)\}'(x) \\ = \; & \frac{u_0^{(k+1)}(0)}{(k+1)!} \{x^{k+1} - P_\Theta x^{k+1}\}'(x) + [R_{k+1}(Eu_0)]'(x) - [P_\Theta R_{k+1}(Eu_0)]'(x) \\ = \; & \frac{u_0^{(k+1)}(0)}{(k+1)!} {\xi_{k+1}^{\Theta}}'(x) + [R_{k+1}(Eu_0)]'(x) - [P_\Theta R_{k+1}(Eu_0)]'(x) \\ = \; & \frac{u_0^{(k+1)}(0)}{(k+1)!} {\xi_{k+1}^{h}}'(x) + [R_{k+1}(Eu_0)]'(x) - [P_\Theta R_{k+1}(Eu_0)]'(x). \end{aligned} \tag{5.77}$$

Thus from (5.71), (5.77), and using $e_h(x) \equiv [u_0 - u_h](x)$, we get for $x \in B_H$,

$$\begin{aligned} e_h{}'(x) \; - \; & \frac{u_0^{(k+1)}(0)}{(k+1)!} {\xi_{k+1}^{\Theta}}'(x) \\ = \; & [R_{k+1}(Eu_0)]'(x) - [P_\Theta R_{k+1}(Eu_0)]'(x) - \delta_h' + \rho_h'. \end{aligned} \tag{5.78}$$

2. From (5.75), we have

$$[R_{k+1}(Eu_0)]'(x) = \frac{1}{k!}\int_0^x (x-t)^k (Eu_0)^{(k+2)}(t)\,dt,$$

and since, $\|u_0\|_{W_\infty^{k+2}(B_{2H})} \leq C$, we have for $x \in B_{2H}$,

$$\begin{aligned}\int_{B_H} |[R_{k+1}(Eu_0)]'|^2\,dx &\leq \int_{B_{2H}} g|[R_{k+1}(Eu_0)]'|^2\,dx \\ &\leq CH^{2k+2}H|u_0|^2_{W_\infty^{k+2}(B_{2H})}.\end{aligned} \tag{5.79}$$

Similarly, again from (5.75), we get

$$\begin{aligned}\int_{B_H} |R_{k+1}(Eu_0)|^2\,dx &\leq \int_{B_{2H}} g|R_{k+1}(Eu_0)|^2\,dx \\ &\leq CH^{2k+4}H|u_0|^2_{W_\infty^{k+2}(B_{2H})}.\end{aligned} \tag{5.80}$$

3. It can be shown from the definition of $g(x)$ that, for $0 \leq j \leq k+1$,

$$\int_{2H}^{\infty} gx^{2j}\,dx = \int_{2H}^{\infty} e^{-\alpha(x-H)}x^{2j}\,dx \leq Ce^{-\alpha H}, \tag{5.81}$$

where C depends on $k+1$. Now, from (5.65) we get

$$\int_{B_H} |[P_\Theta R_{k+1}(Eu_0)]'|^2\,dx \leq \|P_\Theta R_{k+1}(Eu_0)\|^2_{1,g} \leq C\|R_{k+1}(Eu_0)\|^2_{1,g}. \tag{5.82}$$

We note that, from (5.74), we have

$$[R_{k+1}(Eu_0)]'(x) = (Eu_0)'(x) - \sum_{j=0}^{k} \frac{u_0^{(j+1)}(0)}{(j+1)!}x^j.$$

Therefore, using (5.81) and the fact that

$$\int_{2H}^{\infty} g|(Eu_0)'|^2\,dx \leq e^{-\alpha H}\int_{2H}^{\infty} |(Eu_0)'|^2\,dx \leq e^{-\alpha H}|Eu_0|^2_{H^1(\mathbb{R})},$$

we have

$$\begin{aligned}&\int_{2H}^{\infty} g|[R_{k+1}(Eu_0)]'|^2\,dx \\ \leq\ & C\int_{2H}^{\infty} g|(Eu_0)'|^2\,dx + C\sum_{j=0}^{k}\left(\frac{u_0^{(j+1)}(0)}{(j+1)!}\right)^2\int_{2H}^{\infty} gx^{2j}\,dx \\ \leq\ & Ce^{-\alpha H}\{|Eu_0|^2_{H^1(\mathbb{R})} + C\|u_0\|^2_{W_\infty^{k+1}(B_{2H})}\} \\ \leq\ & Ce^{-\alpha H}\{\|u_0\|^2_{H^1(\Omega)} + C\|u_0\|^2_{W_\infty^{k+2}(B_{2H})}\},\end{aligned} \tag{5.83}$$

where C depends on k. Similarly, we can show that

$$\int_{-\infty}^{-2H} g|[R_{k+1}(Eu_0)]'|^2\,dx \leq Ce^{-\alpha H}\{\|u_0\|^2_{H^1(\Omega)} + C\|u_0\|^2_{W^{k+2}_\infty(B_{2H})}\},$$

which, together with (5.83) imply that

$$\int_{\mathbb{R}-B_{2H}} g|[R_{k+1}(Eu_0)]'|^2\,dx \leq Ce^{-\alpha H}\{\|u_0\|^2_{H^1(\Omega)} + C\|u_0\|^2_{W^{k+2}_\infty(B_{2H})}\}. \quad (5.84)$$

Using similar arguments, we can show that

$$\int_{\mathbb{R}-B_{2H}} g|R_{k+1}(Eu_0)|^2\,dx \leq Ce^{-\alpha H}\{\|u_0\|^2_{L_2(\Omega)} + C\|u_0\|^2_{W^{k+2}_\infty(B_{2H})}\}. \quad (5.85)$$

Now combining (5.79), (5.80), (5.82), (5.84), and (5.85) we get

$$\begin{aligned}
&\int_{B_H} |[P_\Theta R_{k+1}(Eu_0)]'|^2\,dx \\
&\quad \leq C\|R_{k+1}(Eu_0)\|^2_{1,g} \\
&\quad = C\int_{\mathbb{R}} g|[R_{k+1}(Eu_0)]'|^2\,dx + C\bar{\Theta}\int_{\mathbb{R}} g|R_{k+1}(Eu_0)|^2\,dx \\
&\quad \leq C(1+\bar{\Theta}H^2)H^{2k+2}H|u_0|^2_{W^{k+2}_\infty(B_{2H})} \\
&\qquad +C(1+\bar{\Theta})e^{-\alpha H}\{\|u_0\|^2_{H^1(\Omega)} + C\|u_0\|^2_{W^{k+2}_\infty(B_{2H})}\}. \qquad (5.86)
\end{aligned}$$

4. We first note from (5.72), that

$$\int_{B_H} {\delta_h'}^2\,dx \leq \|\delta_h\|^2_{1,g} = \|Eu_h - P_\Theta(Eu_h)\|^2_{1,g}. \quad (5.87)$$

Let $P_\Theta(Eu_h) = \tilde{\mathcal{I}}_h^* Eu_h + \mathcal{E}$. Then $\mathcal{E} \in S_h$. Now from Lemma 5.5 and the definition of P_Θ, we have for all $v \in S_h$,

$$\begin{aligned}
B_\Theta(\mathcal{E}, v) &= B_\Theta(P_\Theta(Eu_h) - \tilde{\mathcal{I}}_h^* Eu_h, v) \\
&= B_\Theta(Eu_h - \tilde{\mathcal{I}}_h^* Eu_h, v) \\
&\leq C\|Eu_h - \tilde{\mathcal{I}}_h^* Eu_h\|_{1,g}\|v\|_{1,g^{-1}},
\end{aligned}$$

and hence from Lemma 5.6, we get

$$\|E\|_{1,g} \leq C\sup_{v\in S_h}\frac{B_\Theta(E,v)}{\|v\|_{1,g^{-1}}} \leq C\|Eu_h - \tilde{\mathcal{I}}_h^* Eu_h\|_{1,g}.$$

Thus,

$$\begin{aligned}
\|Eu_h - P_\Theta(Eu_h)\|_{1,g} &\leq \|Eu_h - \tilde{\mathcal{I}}_h^* Eu_h\|_{1,g} + \|\mathcal{E}\|_{1,g} \\
&\leq C\|Eu_h - \tilde{\mathcal{I}}_h^* Eu_h\|_{1,g}. \qquad (5.88)
\end{aligned}$$

We now estimate the RHS of the above inequality. We first note that $Eu_h(x) = u_h(x)$ for $x \in \Omega$. Consider $\underline{\Omega} \subset \Omega$ such that (see Remark 5.4)

$$B_{2H} \subset \underline{\Omega} \quad \text{and} \quad \tilde{\mathcal{I}}_h^* Eu_h|_{\underline{\Omega}} = Eu_h|_{\underline{\Omega}} = u_h|_{\underline{\Omega}}.$$

Therefore, from Lemma 5.2 and using (5.5),

$$\begin{aligned}
\int_{\mathbb{R}} g[(Eu_h - \tilde{\mathcal{I}}_h^* Eu_h)']^2\,dx &= \int_{\mathbb{R}-\underline{\Omega}} g[(Eu_h - \tilde{\mathcal{I}}_h^* Eu_h)']^2\,dx \\
&\leq e^{-\alpha H}[|Eu_h|^2_{H^1(\mathbb{R})} + |\tilde{\mathcal{I}}_h^* Eu_h|^2_{H^1(\mathbb{R})}] \\
&\leq Ce^{-\alpha H}[|Eu_h|^2_{H^1(\mathbb{R})} + \frac{1}{h^2}\|Eu_h\|^2_{L_2(\mathbb{R})}] \\
&\leq \frac{C}{h^2} e^{-\alpha H}\|u_h\|^2_{H^1(\Omega)} \\
&\leq \frac{C}{h^2} e^{-\alpha H}\|u_0\|^2_{H^1(\Omega)}. \qquad (5.89)
\end{aligned}$$

Similarly, we can show using Remark 5.5 that

$$\int_{\mathbb{R}} g[Eu_h - \tilde{\mathcal{I}}_h^* Eu_h]^2\,dx \leq Ce^{-\alpha H}\|u_0\|^2_{L_2(\Omega)},$$

and thus combining it with (5.89), we get

$$\|Eu_h - \tilde{\mathcal{I}}_h^* Eu_h\|^2_{1,g} \leq \frac{C\bar{\Theta}}{h^2} e^{-\alpha H}\|u_0\|^2_{H^1(\Omega)}.$$

Now from (5.87), (5.88), and above, we get

$$\int_{B_H} {\delta_h'}^2\,dx \leq \frac{C\bar{\Theta}}{h^2} e^{-\alpha H}\|u_0\|^2_{H^1(\Omega)}. \qquad (5.90)$$

5. We first note from (5.73) that

$$\int_{B_H(0)} {\rho_h'}^2\,dx \leq \|\rho_h\|^2_{1,g} = \|P_\Theta(Eu_0) - P_\Theta(Eu_h)\|^2_{1,g}. \qquad (5.91)$$

Now using (5.4), we have for all $v \in S_h$,

$$\begin{aligned}
& B_\Theta(\rho_h, v) \\
=\; & B_\Theta(P_\Theta Eu_0 - P_\Theta Eu_h, v) \\
=\; & B_\Theta(Eu_0 - Eu_h, v) \\
=\; & B^\Omega(Eu_0 - Eu_h, v) + B^{\mathbb{R}-\Omega}(Eu_0 - Eu_h, v) + \Theta D^{\mathbb{R}}(Eu_0 - Eu_h, v) \\
=\; & B^\Omega(u_0 - u_h, v) + B^{\mathbb{R}-\Omega}(Eu_0 - Eu_h, v) + \Theta D^{\mathbb{R}}(Eu_0 - Eu_h, v) \\
=\; & B^{\mathbb{R}-\Omega}(Eu_0 - Eu_h, v) + \Theta D^{\mathbb{R}-\Omega}(Eu_0 - Eu_h, v) + \Theta D^{\Omega}(Eu_0 - Eu_h, v) \\
=\; & B_\Theta^{\mathbb{R}-\Omega}(Eu_0 - Eu_h, v) + \Theta D^{\Omega}(u_0 - u_h, v). \qquad (5.92)
\end{aligned}$$

Also, for $v \in S_h$,

$$\begin{aligned}
& B_{\Theta}^{\mathbb{R}-\Omega}(Eu_0 - Eu_h, v) \\
= \; & \int_{\mathbb{R}-\Omega} [(Eu_0 - Eu_h)'v' + \bar{\Theta}(Eu_0 - Eu_h)v]\, dx \\
\leq \; & C|||Eu_0 - Eu_h|||_{1,g,\mathbb{R}-\Omega}\, |||v|||_{1,g^{-1},\mathbb{R}-\Omega} \\
\leq \; & C|||Eu_0 - Eu_h|||_{1,g,\mathbb{R}-\Omega}\, \|v\|_{1,g^{-1}}, \qquad (5.93)
\end{aligned}$$

where

$$\begin{aligned}
|||v|||_{1,g^{-1},\mathbb{R}-\Omega}^2 &= \int_{\mathbb{R}-\Omega} g^{-1}{v'}^2\, dx + \bar{\Theta}\int_{\mathbb{R}-\Omega} g^{-1}v^2\, dx; \\
|||Eu_0 - Eu_h|||_{1,g,\mathbb{R}-\Omega}^2 &= \int_{\mathbb{R}-\Omega} g(Eu_0 - Eu_h)'^2\, dx \\
&\quad + \bar{\Theta}\int_{\mathbb{R}-\Omega} g(Eu_0 - Eu_h)^2\, dx.
\end{aligned}$$

From the definition of $g(x)$, we can show that

$$\begin{aligned}
|||Eu_0 - Eu_h|||_{1,g,\mathbb{R}-\Omega}^2 &\leq e^{-\alpha H}\bar{\Theta}\|Eu_0 - Eu_h\|_{H^1(\mathbb{R}-\Omega)}^2 \\
&\leq Ce^{-\alpha H}\bar{\Theta}\|u_0 - u_h\|_{H^1(\Omega)}^2 \\
&\leq Ce^{-\alpha H}\bar{\Theta}\|u_0\|_{H^1(\Omega)}^2. \qquad (5.94)
\end{aligned}$$

Now using the definition of $g(x)$ and (5.6) with $R = 2H$, we get

$$\begin{aligned}
\int_{\Omega} g(u_0 - u_h)^2\, dx &= \int_{B_{2H}} g(u_0 - u_h)^2\, dx + \int_{\Omega - B_{2H}} g(u_0 - u_h)^2\, dx \\
&\leq \|u_0 - u_h\|_{L_2(B_{2H})}^2 + e^{-\alpha H}\|u_0 - u_h\|_{L_2(\Omega)}^2 \\
&\leq Ch^{2k+2}H\|u_0\|_{H^{k+1}(\Omega)}^2 + e^{-\alpha H}\|u_0\|_{H^1(\Omega)}^2,
\end{aligned}$$

and therefore,

$$\begin{aligned}
& \frac{\Theta}{\|v\|_{1,g^{-1}}} D^{\Omega}(u_0 - u_h, v) \\
= \; & \frac{\Theta}{\|v\|_{1,g^{-1}}} \int_{\Omega} (u_0 - u_h)v\, dx \\
\leq \; & \frac{\Theta}{\|v\|_{1,g^{-1}}} \left(\int_{\Omega} g(u_0 - u_h)^2\, dx\right)^{1/2} \left(\int_{\Omega} g^{-1}v^2\, dx\right)^{1/2} \\
\leq \; & \Theta^{\frac{1}{2}}Ch^{k+1}H^{\frac{1}{2}}\|u_0\|_{H^{k+1}(\Omega)} + \Theta^{\frac{1}{2}}e^{-\alpha H/2}\|u_0\|_{H^1(\Omega)}. \qquad (5.95)
\end{aligned}$$

Now, from the inf-sup condition (5.46) and using (5.92) and (5.93), we have

$$\begin{aligned}
\|\rho_h\|_{1,g} &\leq C\sup_{v \in S_h} \frac{B_{\Theta}(\rho_h, v)}{\|v\|_{1,g^{-1}}} \\
&\leq C|||Eu_0 - Eu_h|||_{1,g,\mathbb{R}-\Omega} + \sup_{v \in S_h} \frac{\Theta}{\|v\|_{1,g^{-1}}} D^{\Omega}(u_0 - u_h, v),
\end{aligned}$$

and thus, using (5.91), (5.94) and (5.95), we have

$$\begin{aligned}\int_{B_H} {\rho_h'}^2\,dx &\le \|\rho_h\|_{1,g}^2 \\ &\le Ce^{-\alpha H}\bar{\Theta}\|u_0\|^2_{H^1(\Omega)} + C\Theta h^{2k+2}H\|u_0\|^2_{H^{k+1}(\Omega)}. \quad (5.96)\end{aligned}$$

6. We first note from (5.69) that ${\xi^{\Theta}_{k+1}}'(x) = {\xi^h_{k+1}}'(x)$ where ξ^h_{k+1} is defined in (4.36). Let $T(u_0) \equiv \frac{u_0^{(k+1)}(0)}{(k+1)!}$. Then from (5.78), we have

$$\begin{aligned}&{e_h}'(x) - T(u_0){\xi^h_{k+1}}'(x) \\ &\qquad = [R_{k+1}(Eu_0)]'(x) - [P_\Theta R_{k+1}(Eu_0)]'(x) - \delta_h' + \rho_h',\end{aligned}$$

and therefore, from (5.79), (5.86), (5.90), and (5.96), we have

$$\begin{aligned}&\int_{B_H}\left({e_h}' - T(u_0){\xi^h_{k+1}}'\right)^2\,dx \\ \le\ & C\int_{B_H} |[R_{k+1}(Eu_0)]'|^2\,dx + C\int_{B_H} |[P_\Theta R_{k+1}(Eu_0)]'|^2\,dx \\ &\qquad\qquad + C\int_{B_H} {\delta_h'}^2\,dx + C\int_{B_H}\rho_h^2\,dx \\ \le\ & CH^{2k+2}H|u_0|^2_{W^{k+2}_\infty(B_{2H})} + C(1+\bar{\Theta}H^2)H^{2k+2}H|u_0|^2_{W^{k+2}_\infty(B_{2H})} \\ & + C(1+\bar{\Theta})e^{-\alpha H}\{\|u_0\|^2_{H^1(\Omega)} + C\|u_0\|^2_{W^{k+2}_\infty(B_{2H})}\} + \frac{C\bar{\Theta}}{h^2}e^{-\alpha H}\|u_0\|^2_{H^1(\Omega)} \\ & + Ce^{-\alpha H}\bar{\Theta}\|u_0\|^2_{H^1(\Omega)} + C\Theta h^{2k+2}H\|u_0\|^2_{H^{k+1}(\Omega)} \\ \le\ & C\left[H^{2k+2}H + (1+\bar{\Theta}H^2)H^{2k+2}H + (1+\bar{\Theta})e^{-\alpha H}\right. \\ &\qquad\qquad \left. + \frac{\bar{\Theta}}{h^2}e^{-\alpha H} + \Theta h^{2k+2}H\right]M(u_0), \qquad (5.97)\end{aligned}$$

where

$$M(u_0) = \|u_0\|^2_{H^{k+1}(\Omega)} + \|u_0\|^2_{W^{k+2}_\infty(B_{2H})}$$

We will now choose α, Θ, and H, where $H = h^\gamma$ and $\gamma < 1$. First we choose γ such that

$$H^{k+2} = h^{k+1}, \qquad (5.98)$$

which implies that

$$\begin{aligned} h^{\gamma(k+2)} &= h^{k+1} \\ \text{or, } \gamma(k+2) &= k+1 \\ \text{or, } \gamma &= \frac{k+1}{k+2} < 1.\end{aligned}$$

Let $\epsilon > 0$, which depends on γ, be such that $\epsilon^* \equiv 1 - \gamma - \epsilon > 0$. We will now choose α such that

$$e^{-\alpha H} \le h^{2k+2}h^2h^{2\gamma+2\epsilon}H = h^{2k+4+3\gamma+2\epsilon}. \qquad (5.99)$$

This implies that

$$\alpha \geq C_1(\ln h^{-1})h^{-\gamma},$$

where $C_1 = 2k+4+3\gamma+2\epsilon$. Since $h^{-\epsilon} > \ln h^{-1}$ for small h, we take

$$\alpha \equiv C_1 h^{-(\gamma+\epsilon)}. \tag{5.100}$$

We now choose

$$\bar{\Theta} \equiv (C_2)^2 h^{-2(\gamma+\epsilon)}, \quad C_2 > C_1. \tag{5.101}$$

We note from (5.100) that $\alpha h = C_1 h^{1-\gamma-\epsilon} = C_1 h^{\epsilon^*} < 1$ for small h, and $\lim_{h\to 0} \alpha h = 0$. Thus αh can be made sufficiently small; this was one of the assumptions in Lemma 5.7. Also by choosing C_2 large enough in (5.101), we can make $\frac{\alpha^2}{\bar{\Theta}} = (C_1/C_2)^2 << 1$, *i.e.,* sufficiently small, which was another assumption in Lemma 5.7. Thus the conclusion of Lemma 5.7 is true for the choices of α and $\bar{\Theta}$ given in (5.100) and (5.101), respectively.

Now, for these choices of γ, α, and $\bar{\Theta}$, we have

$$\bar{\Theta} h^{2k+2} \quad = \quad \bar{\Theta} h^{2(\gamma+\epsilon)} h^{2k} h^{2(1-\gamma-\epsilon)} = C_2^2 h^{2k+2\epsilon^*}. \tag{5.102}$$

Using (5.98) and (5.102) we have

$$\bar{\Theta} H^2 H^{2k+2} \quad = \quad \bar{\Theta} H^{2k+4} = \bar{\Theta} h^{2k+2} = C_2^2 h^{2k+2\epsilon^*}. \tag{5.103}$$

Also from (5.99), we get

$$\bar{\Theta} e^{-\alpha H} \quad \leq \quad h^{2k+4} H \bar{\Theta} h^{2\gamma+2\epsilon} \leq C_2^2 h^{2k+2} H, \tag{5.104}$$

and

$$\frac{\bar{\Theta}}{h^2} e^{-\alpha H} \quad \leq \quad h^{2k+2} H \bar{\Theta} h^{2\gamma+2\epsilon} = C_2^2 h^{2k+2} H. \tag{5.105}$$

Thus, using (5.98),(5.102)–(5.105) in (5.97), we get

$$\|{e_h}' - T(u_0){\xi_{k+1}^h}'\|_{L_2(B_H)} \leq C h^{k+\epsilon^*} H^{1/2} M(u_0)^{\frac{1}{2}}, \tag{5.106}$$

and hence using (5.7) with $\rho = H$, we have

$$\frac{\|{e_h}' - T(u_0){\xi_{k+1}^h}'\|_{L_2(B_H)}}{\|{e_h}'\|_{L_2(B_H)}} \leq C h^{\epsilon^*}$$

where $M(u_0)^{\frac{1}{2}}/\|u_0\|_{H^{k+1}(\Omega)} \leq C$, which is the desired result. □

Remark 5.9 The balancing of various terms in item no. 6 of the proof of Theorem 5.1 is similar to balancing used in the superconvergence of FEM solution (see [19].

Remark 5.10 Assuming that our superconvergence result is valid in L_∞, *i.e.*, assuming that for $x \in B_H$, there exists $\epsilon^* > 0$, such that

$$e_h'(x) = A(u_0)h^k {\xi_{k+1}}'(\frac{x}{h}) + O(h^{k+\epsilon^*}),$$

we see that the zeros of ${\xi_{k+1}}'(\frac{x}{h})$ are the superconvergence points. In Figure 5.1,

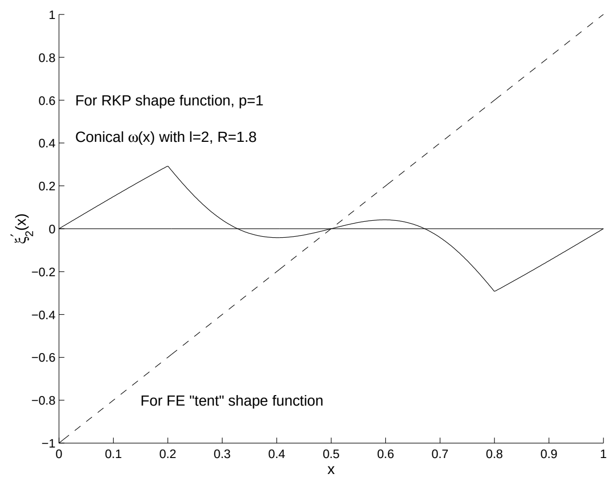

Figure 5.1: The plot of ${\xi_2}'(y),\ 0 \le y \le 1$ for (a) RKP shape functions, reproducing of order $k = 1$, corresponding to the conical weight function with $l = 2,\ R = 1.8$ (b) standard "tent" functions used in FEM.

we have presented the plot of ${\xi_{k+1}}'(y)$ for the RKP shape functions, reproducing of order $k = 1$, with respect to the weight function $w(x)$ given by (4.4) with $l = 2$ in 1-d. We have also included the plot of ${\xi_{k+1}}'(y),\ k = 1$ (the dashed curve) for the standard tent functions that are used as shape in FEM. We note that ${\xi_2}'(y)$ for the tent function has only one zero, where as ${\xi_2}'(y)$ has 5 zeros. Thus the superconvergence points, for the RKP shape function could be distributed quite differently than the corresponding points for standard tent functions in FEM.

6 The Generalized Finite Element Method

The idea of the Generalized Finite Element Method (GFEM) was first introduced in [16] to address elliptic problems with rough coefficients. This idea was later extended, and called the Partition of Unity Method (PUM) in [17] and [62]. In the current literature, PUM is referred to as GFEM ([82],[83]). In this section, we will first describe the GFEM and present the relevant approximation results. We will then discuss the selection of an optimal or near optimal approximating space, to be used in the GFEM, in certain situations.

6.1 Description of GFEM and Related Approximation Results

In this section we will discuss the GFEM in the context of general particle-shape function systems, which were discussed in Section 3.3. Suppose u_0 is the solution of our model problem (2.1), (2.2) (or (2.3)). We consider a family $\{\mathcal{M}^\nu\}_{\nu\in N}$ of particle shape function systems satisfying assumptions A1–A7 with $k=0$ and $\mathcal{A}^\nu_{\underline{x}} = I$; assumption A5 then reads

$$\sum_{\underline{x}\in X^\nu} \phi^\nu_{\underline{x}}(x) = 1, \quad \text{for all } x \in \mathbb{R}^n. \tag{6.1}$$

The partition of unity (6.1) is the starting point of GFEM. We will need additional assumptions on $\{\mathcal{M}^\nu\}_{\nu\in N}$, namely,

$$\|\phi^\nu_{\underline{x}}\|_{L^\infty(\mathbb{R}^n)} \leq C_1 \tag{6.2}$$

and

$$\|\nabla\phi^\nu_{\underline{x}}\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_2}{\operatorname{diam}(\eta^\nu_{\underline{x}})}, \tag{6.3}$$

for all $\underline{x} \in X^\nu$, and all $\nu \in N$. In (6.3), we implicitly assume that $q > n/2$. We also assume that there is a constant C such that

$$\operatorname{diam}(\eta^\nu_{\underline{x}}) \leq C, \text{ for all } \underline{x} \in X^\nu \text{ and for all } \nu.$$

For each $\underline{x} \in X^\nu$, we assume that we have a finite dimensional space $V^\nu_{\underline{x}}$ of functions that has good approximation properties. We refer to $V^\nu_{\underline{x}}$ as local approximating spaces. We define a set of particles A^ν_Ω, namely,

$$A^\nu_\Omega = \{\underline{x} \in X^\nu : \mathring{\eta}^\nu_{\underline{x}} \cap \Omega \neq \emptyset\}, \tag{6.4}$$

for each $\nu \in N$. From (6.1) we have

$$\sum_{\underline{x}\in A^\nu_\Omega} \phi^\nu_{\underline{x}}(x) = 1, \quad \text{for all } x \in \Omega. \tag{6.5}$$

For an approximating space on Ω, we then consider

$$V^\nu = \{v\big|_\Omega : v = \sum_{\underline{x}\in A^\nu_\Omega} \phi^\nu_{\underline{x}}\psi^\nu_{\underline{x}}, \text{ where } \psi^\nu_{\underline{x}} \in V^\nu_{\underline{x}}\}. \tag{6.6}$$

The GFEM is the Galerkin method (2.7) with $\tilde{B} = B$ and $S = V^\nu$, and we will denote the approximate solution u_S, obtained from GFEM, by u_{GFEM}. When GFEM is used to approximate the solution u_0 of the Neumann problem, $V^\nu_{\underline{x}}$ can be any finite dimensional subspace of $H^1(\mathring{\eta}^\nu_{\underline{x}})$. But, when GFEM is used to approximate the solution u_0 of the Dirichlet problem, with the boundary condition (2.3), the functions in $V^\nu_{\underline{x}}$ are required to satisfy $v|_{\mathring{\eta}^\nu_{\underline{x}}\cap\partial\Omega} = 0$, for particles $\underline{x}$ for which $|\mathring{\eta}^\nu_{\underline{x}} \cap \Omega| > 0$. Thus the approximating space $V^\nu \subset H^1_0(\Omega)$.

Our next theorem states an approximation result for V^ν. We will follow the ideas presented in [15, 16, 17, 62, 82, 83].

Theorem 6.1 *Suppose $u \in H^1(\Omega)$ and suppose, for all $\underline{x} \in A^\nu_\Omega$, there exists $\psi^\nu_{\underline{x}} \in V^\nu_{\underline{x}}$ such that*

$$\|u - \psi_{\underline{x}}\|_{L_2(\eta^\nu_{\underline{x}} \cap \Omega)} \leq \epsilon_1(\underline{x}), \tag{6.7}$$

$$\|\nabla(u - \psi_{\underline{x}})\|_{L_2(\eta^\nu_{\underline{x}} \cap \Omega)} \leq \epsilon_2(\underline{x}). \tag{6.8}$$

Then the function

$$u_{ap} = \sum_{\underline{x} \in A^\nu_\Omega} \phi^\nu_{\underline{x}} \psi^\nu_{\underline{x}} \in V^\nu \tag{6.9}$$

satisfies

$$\|u - u_{ap}\|_{L_2(\Omega)} \leq \kappa^{1/2} C_1 \Big(\sum_{\underline{x} \in A^\nu_\Omega} \epsilon_1^2(\underline{x}) \Big)^{1/2} \tag{6.10}$$

and

$$\|\nabla(u - u_{ap})\|_{L_2(\Omega)} \leq (2\kappa)^{1/2} \Big(\sum_{\underline{x} \in A^\nu_\Omega} \big(\frac{C_2}{diam(\eta^\nu_{\underline{x}})}\big)^2 \epsilon_1^2(\underline{x}) + C_1^2 \epsilon_2^2(\underline{x}) \Big)^{1/2}. \tag{6.11}$$

Proof. We will prove only (6.11), since (6.10) can be proved similarly.

Since $\phi^\nu_{\underline{x}}$, for $\underline{x} \in A^\nu_\Omega$, form a partition of unity for Ω (see (6.5)), we have

$$\begin{aligned}
&\|\nabla(u - u_{ap})\|^2_{L_2(\Omega)} \\
&\quad = \|\nabla \sum_{\underline{x} \in A^\nu_\Omega} \phi^\nu_{\underline{x}} (u - \psi_{\underline{x}})\|^2_{L_2(\Omega)} \\
&\quad \leq 2 \| \sum_{\underline{x} \in A^\nu_\Omega} (u - \psi_{\underline{x}}) \nabla \phi^\nu_{\underline{x}} \|^2_{L_2(\Omega)} + 2 \| \sum_{\underline{x} \in A^\nu_\Omega} \phi^\nu_{\underline{x}} \nabla(u - \psi_{\underline{x}}) \|^2_{L_2(\Omega)}.
\end{aligned} \tag{6.12}$$

For any $x \in \Omega$, the sums $\sum_{\underline{x} \in A^\nu_\Omega} (u - \psi_{\underline{x}}) \nabla \phi^\nu_{\underline{x}}$ and $\sum_{\underline{x} \in A^\nu_\Omega} \phi^\nu_{\underline{x}} \nabla (u - \psi_{\underline{x}})$ have at most κ non-zero terms (see Remark 3.4 and (3.61)). Therefore,

$$| \sum_{\underline{x} \in A^\nu_\Omega} (u - \psi_{\underline{x}}) \nabla \phi^\nu_{\underline{x}} |^2 \leq \kappa \sum_{\underline{x} \in A^\nu_\Omega} |(u - \psi_{\underline{x}}) \nabla \phi^\nu_{\underline{x}}|^2,$$

and

$$| \sum_{\underline{x} \in A^\nu_\Omega} \phi^\nu_{\underline{x}} \nabla (u - \psi_{\underline{x}}) |^2 \leq \kappa \sum_{\underline{x} \in A^\nu_\Omega} |\phi^\nu_{\underline{x}} \nabla (u - \psi_{\underline{x}})|^2.$$

Hence, from (6.12), (6.7), (6.8), recalling that $\text{supp}(\phi^\nu_{\underline{x}}) = \eta^\nu_{\underline{x}}$, we have

$$\begin{aligned}
&\|\nabla(u - u_{ap})\|^2_{L_2(\Omega)} \\
&\quad \leq 2\kappa \sum_{\underline{x} \in A^\nu_\Omega} \|(u - \psi_{\underline{x}}) \nabla \phi^\nu_{\underline{x}}\|^2_{L_2(\Omega)} + 2\kappa \sum_{\underline{x} \in A^\nu_\Omega} \|\phi^\nu_{\underline{x}} \nabla (u - \psi_{\underline{x}})\|^2_{L_2(\Omega)} \\
&\quad = 2\kappa \sum_{\underline{x} \in A^\nu_\Omega} \|(u - \psi_{\underline{x}}) \nabla \phi^\nu_{\underline{x}}\|^2_{L_2(\Omega \cap \eta^\nu_{\underline{x}})} + 2\kappa \sum_{\underline{x} \in A^\nu_\Omega} \|\phi^\nu_{\underline{x}} \nabla (u - \psi_{\underline{x}})\|^2_{L_2(\Omega \cap \eta^\nu_{\underline{x}})} \\
&\quad \leq 2\kappa \sum_{\underline{x} \in A^\nu_\Omega} \Big(\big(\frac{C_2}{\text{diam}(\eta^\nu_{\underline{x}})}\big)^2 \epsilon_1^2(\underline{x}) + C_1^2 \epsilon_2^2(\underline{x}) \Big),
\end{aligned}$$

which is the desired result. □

Remark 6.1 We note that $\epsilon_1(\underline{x})$, $\epsilon_2(\underline{x})$ in (6.7), (6.8) depend on the parameter ν.

We will now show that both the terms of the estimate (6.11) are of the same order with additional assumptions on V^ν. These additional assumptions depend on the boundary conditions of the approximated function.

Theorem 6.2 *Suppose $u_0 \in H^1(\Omega)$ is the solution of the Neumann problem (2.1), (2.2), and suppose there exists $\psi^\nu_{\underline{x}} \in V^\nu_{\underline{x}}$, $\underline{x} \in A^\nu_\Omega$, such that (6.7) and (6.8) are satisfied. Moreover, assume that for $\underline{x} \in A^h_\Omega$, the space $V^\nu_{\underline{x}}$ contains constant functions and that*

$$\inf_{\lambda\in\mathbb{R}} \|v-\lambda\|_{L_2(\eta^\nu_{\underline{x}}\cap\Omega)} \le C\ (diam(\eta^\nu_{\underline{x}}))\ \|\nabla v\|_{L_2(\eta^\nu_{\underline{x}}\cap\Omega)}, \quad \textit{for all } v\in H^1(\eta^\nu_{\underline{x}}\cap\Omega), \tag{6.13}$$

where C is independent of $\underline{x} \in X^\nu$ and ν. Then there exists $\tilde{\psi}^\nu_{\underline{x}} \in V^\nu_{\underline{x}}$ so that the corresponding function,

$$\tilde{u}_{ap} = \sum_{\underline{x}\in A^\nu_\Omega} \phi^\nu_{\underline{x}} \tilde{\psi}^\nu_{\underline{x}} \in V^\nu,$$

satisfies

$$\|u_0 - \tilde{u}_{ap}\|_{H^1(\Omega)} \le C\Big(\sum_{\underline{x}\in A^h_\Omega} \epsilon_2^2(\underline{x})\Big)^{1/2}, \tag{6.14}$$

where C is independent of u_0 and ν.

Proof. Let $\psi^\nu_{\underline{x}} \in V^\nu_{\underline{x}}$, $\underline{x} \in A^\nu_\Omega$, satisfy (6.7) and (6.8). Define $\tilde{\psi}^\nu_{\underline{x}} = \psi^\nu_{\underline{x}} + r^\nu_{\underline{x}}$, where $r^\nu_{\underline{x}} \in \mathbb{R}$ satisfies

$$\|u_0 - \tilde{\psi}^\nu_{\underline{x}}\|_{L_2(\eta^\nu_{\underline{x}}\cap\Omega)} = \inf_{\lambda\in\mathbb{R}} \|u_0 - \psi^\nu_{\underline{x}} - \lambda\|_{L_2(\eta^\nu_{\underline{x}}\cap\Omega)}. \tag{6.15}$$

Since $V^\nu_{\underline{x}}$ contains constant functions, it is clear that $\tilde{\psi}_{\underline{x}} \in V_{\underline{x}}$. Also, from (6.15), (6.13) with $v = u_0 - \psi^\nu_{\underline{x}}$, and (6.8), we have

$$\begin{aligned} \|u - \tilde{\psi}_{\underline{x}}\|_{L_2(\eta^\nu_{\underline{x}}\cap\Omega)} &\le C\ \mathrm{diam}(\eta^\nu_{\underline{x}})\,\|\nabla(u - \psi^\nu_{\underline{x}})\|_{L_2(\eta^\nu_{\underline{x}}\cap\Omega)} \\ &\le C\ \mathrm{diam}(\eta^\nu_{\underline{x}})\,\epsilon_2(\underline{x}). \end{aligned} \tag{6.16}$$

Let $\tilde{u}_{ap} = \sum_{\underline{x}\in A^\nu_\Omega} \phi^\nu_{\underline{x}}\tilde{\psi}_{\underline{x}}$. Recall that $\phi^\nu_{\underline{x}}$, $\underline{x} \in A^\nu_\Omega$, is a partition of unity for Ω. Then, following the arguments in the proof of Theorem 6.1 and using (3.61), (6.2), we can show that

$$\begin{aligned}\|u-\tilde{u}_{ap}\|^2_{L_2(\Omega)} &= \|\sum_{\underline{x}\in A^h_\Omega}\phi^\nu_{\underline{x}}(u-\tilde{\psi}_{\underline{x}})\|^2_{L_2(\Omega)}\\ &\le \kappa\sum_{\underline{x}\in A^h_\Omega}\|\phi^\nu_{\underline{x}}(u-\tilde{\psi}_{\underline{x}})\|^2_{L_2(\Omega)}\\ &= \kappa\sum_{\underline{x}\in A^h_\Omega}\|\phi^\nu_{\underline{x}}(u-\tilde{\psi}_{\underline{x}})\|^2_{L_2(\Omega\cap\eta^\nu_{\underline{x}})}\\ &\le C\sum_{\underline{x}\in A^h_\Omega}\|(u-\tilde{\psi}_{\underline{x}})\|^2_{L_2(\Omega\cap\eta^\nu_{\underline{x}})}, \end{aligned}\tag{6.17}$$

and using (6.16) in this inequality, we get

$$\|u-\tilde{u}_{ap}\|^2_{L_2(\Omega)}\le C\sum_{\underline{x}\in A^h_\Omega}(\mathrm{diam}(\eta^\nu_{\underline{x}}))^2\epsilon^2_2(\underline{x}).\tag{6.18}$$

Again, following the arguments in the proof of Theorem (6.1), and using (6.2), (6.3), we can show that

$$\begin{aligned}&\|\nabla(u-\tilde{u}_{ap})\|^2_{L_2(\Omega)}\\ &\quad\le 2\kappa\sum_{\underline{x}\in A^\nu_\Omega}\|(u-\tilde{\psi}^\nu_{\underline{x}})\nabla\phi^\nu_{\underline{x}}\|^2_{L_2(\Omega\cap\eta^\nu_{\underline{x}})}+2\kappa\sum_{\underline{x}\in A^\nu_\Omega}\|\phi^\nu_{\underline{x}}\nabla(u-\tilde{\psi}^\nu_{\underline{x}})\|^2_{L_2(\Omega\cap\eta^\nu_{\underline{x}})}\\ &\quad\le C\sum_{\underline{x}\in A^h_\Omega}\frac{1}{(\mathrm{diam}(\eta^\nu_{\underline{x}}))^2}\|u-\tilde{\psi}^\nu_{\underline{x}}\|^2_{L_2(\Omega\cap\eta^\nu_{\underline{x}})}\\ &\qquad\qquad+C\sum_{\underline{x}\in A^h_\Omega}\|\nabla(u-\tilde{\psi}^\nu_{\underline{x}})\|^2_{L_2(\Omega\cap\eta^\nu_{\underline{x}})}. \end{aligned}\tag{6.19}$$

By first noting that $\nabla(u-\tilde{\psi}^\nu_{\underline{x}})=\nabla(u-\psi^\nu_{\underline{x}})$, and then using (6.16) and (6.8) in the above inequality, we get

$$\|\nabla(u-\tilde{u}_{ap})\|^2_{L_2(\Omega)}\le C\sum_{\underline{x}\in A^\nu_\Omega}\epsilon^2_2(\underline{x}).\tag{6.20}$$

Combining this with (6.18) we get (6.14), where we used that $\mathrm{diam}(\eta^\nu_{\underline{x}})\le C$ for all $\underline{x}\in X^\nu$ and for all ν. □

Theorem 6.3 *Suppose $u_0\in H^1_0(\Omega)$ is the solution of the Dirichlet problem (2.1), (2.3), and suppose $V^\nu_{\underline{x}}$, $\underline{x}\in A^\nu_\Omega$, satisfy the following assumptions:*

(a) For all $\underline{x}\in A^h_\Omega$ such that $\eta^\nu_{\underline{x}}\cap\partial\Omega=\emptyset$, $V^\nu_{\underline{x}}$ contains constant functions, and (6.7), (6.8), and (6.13) hold.

(b) For all $\underline{x}\in A^h_\Omega$ such that $|\eta^\nu_{\underline{x}}\cap\partial\Omega|>0$, functions $v\in V^\nu_{\underline{x}}$ satisfy $v\big|_{\eta^\nu_{\underline{x}}\cap\partial\Omega}=0$, and there is a constant C, independent of $\underline{x}$ and ν, such that

$$\|v\|_{L_2(\eta^\nu_{\underline{x}}\cap\Omega)}\le C\ (diam(\eta^\nu_{\underline{x}}))\ \|\nabla v\|_{L_2(\eta^\nu_{\underline{x}}\cap\Omega)},\tag{6.21}$$

for all $v \in H^1(\eta^\nu_{\underline{x}} \cap \Omega)$ *satisfying* $v = 0$ *on* $\partial\Omega$*. Moreover (6.7) and (6.8) hold for* u *satisfying* $u\big|_{\eta^\nu_{\underline{x}} \cap \partial\Omega} = 0$*.*

Then there exists $\tilde{\psi}^\nu_{\underline{x}} \in V^\nu_{\underline{x}}$ *so that the corresponding function,*

$$\tilde{u}_{ap} = \sum_{\underline{x} \in A^\nu_\Omega} \phi^\nu_{\underline{x}} \tilde{\psi}^\nu_{\underline{x}} \in V^\nu,$$

satisfies

$$\|u_0 - \tilde{u}_{ap}\|_{H^1(\Omega)} \leq C\Big(\sum_{\underline{x} \in A^h_\Omega} \epsilon_2^2(\underline{x})\Big)^{1/2}, \tag{6.22}$$

where C *is independent of* u_0 *and* ν*.*

Proof. We first divide the set A^ν_ω into two disjoint sets, namely,

$$\begin{aligned} A^\nu_{\Omega,I} &= \{\underline{x} \in A_\Omega : \eta^\nu_{\underline{x}} \cap \partial\Omega = \emptyset\}, \text{ and} \\ A^\nu_{\Omega,B} &= \{\underline{x} \in A_\Omega : \eta^\nu_{\underline{x}} \cap \partial\Omega \neq \emptyset\}. \end{aligned}$$

Let $\psi^\nu_{\underline{x}} \in V^\nu_{\underline{x}}$, $\underline{x} \in A^\nu_\Omega$, satisfy (6.7) and (6.8). Define $\tilde{\psi}^\nu_{\underline{x}}$, for $\underline{x} \in A^\nu_{\Omega,I}$, as in the proof of Theorem 6.2. We know from assumption (a) that, for $\underline{x} \in A^\nu_{\Omega,I}$, (6.13) holds and $V^\nu_{\underline{x}}$ contains constant functions. Therefore following the argument leading to (6.16) in Theorem 6.2, we get

$$\|u_0 - \tilde{\psi}_{\underline{x}}\|_{L_2(\eta^\nu_{\underline{x}} \cap \Omega)} \leq C \mathrm{diam}(\eta^\nu_{\underline{x}})\, \epsilon_2(\underline{x}), \quad \underline{x} \in A^\nu_{\Omega,I}. \tag{6.23}$$

For $\underline{x} \in A^\nu_{\Omega,B}$, we set $\tilde{\psi}^\nu_{\underline{x}} = \psi^\nu_{\underline{x}}$. Now, $u_0\big|_{\eta^\nu_{\underline{x}} \cap \partial\Omega} = 0$ and from assumption (b), we know that $\psi^\nu_{\underline{x}}\big|_{\eta^\nu_{\underline{x}} \cap \partial\Omega} = 0$ for $\underline{x} \in A^\nu_{\Omega,B}$. Thus, using (6.21), with $v = u_0 - \psi^\nu_{\underline{x}}$, and (6.8), we have

$$\begin{aligned} \|u_0 - \tilde{\psi}_{\underline{x}}\|_{L_2(\eta^\nu_{\underline{x}} \cap \Omega)} &= \|u - \psi_{\underline{x}}\|_{L_2(\eta^\nu_{\underline{x}} \cap \Omega)} \\ &\leq C\, \mathrm{diam}(\eta^\nu_{\underline{x}})\, \epsilon_2(\underline{x}), \quad \underline{x} \in A^\nu_{\Omega,B}. \end{aligned} \tag{6.24}$$

Following the same steps leading to (6.17) in the proof of Theorem 6.2, and using (6.23) and (6.24), we get

$$\begin{aligned} \|u_0 - u_{ap}\|^2_{L_2(\Omega)} &\leq C \sum_{\underline{x} \in A^\nu_\Omega} \|u_0 - \tilde{\psi}^\nu_{\underline{x}}\|^2_{L_2(\eta^\nu_{\underline{x}} \cap \Omega)} \\ &= C \sum_{\underline{x} \in A^\nu_{\Omega,I}} \|u_0 - \tilde{\psi}^\nu_{\underline{x}}\|^2_{L_2(\eta^\nu_{\underline{x}} \cap \Omega)} + C \sum_{\underline{x} \in A^\nu_{\Omega,B}} \|u_0 - \tilde{\psi}^\nu_{\underline{x}}\|^2_{L_2(\eta^\nu_{\underline{x}} \cap \Omega)} \\ &\leq C \sum_{\underline{x} \in A^\nu_\Omega} (\mathrm{diam}(\eta^\nu_{\underline{x}}))^2 \epsilon_2^2(\underline{x}). \end{aligned} \tag{6.25}$$

Similarly, following the steps leading to (6.20) in the proof of Theorem 6.2, we get

$$\|\nabla(u - \tilde{u}_{ap})\|^2_{L_2(\Omega)} \leq C \sum_{\underline{x} \in A^\nu_\Omega} \epsilon_2^2(\underline{x}), \tag{6.26}$$

and combining this with (6.25), we get (6.22), where we used the assumption that $\mathrm{diam}(\eta^\nu_{\underline{x}}) \leq C$ for all $\underline{x} \in X^\nu$ and for all ν. □

Remark 6.2 It is clear from (6.14) and (2.8) that if u_0 is the solution of (2.1), (2.2), then

$$\|u_0 - u_{GFEM}\|_{H^1(\Omega)} \leq C\Big(\sum_{\underline{x}\in X^\nu} \epsilon_2^2(\underline{x})\Big)^{1/2},$$

provided the local approximation spaces $V^\nu_{\underline{x}}$ contain constant functions, and (6.13) holds. The above estimate is also true if u_0 is the solution of (2.1), (2.3) provided conditions (a) and (b) of Theorem 6.3 are satisfied. We note that in the later case, *i.e.,* when u_0 satisfies the Dirichlet boundary condition, $u_0|_{\partial\Omega} = 0$, the space $V^\nu_{\underline{x}}$, corresponding to a particle $\underline{x}$ such that $\eta^\nu_{\underline{x}}$ intersects $\partial\Omega$, does not need to include constant functions, but the functions in $V^\nu_{\underline{x}}$ have to satisfy the Dirichlet boundary condition on $\eta^\nu_{\underline{x}} \cap \partial\Omega$.

Remark 6.3 The conditions (a), (b) in Theorem 6.3, and (6.13) are known as the *uniform Poincaré property.* These conditions put restrictions on the shapes of the $\eta^\nu_{\underline{x}}$'s. For a detailed discussion on this property, see [17].

Remark 6.4 The constant C_2 in (6.3) is related to the ratio of the radius of the largest ball contained in $\eta^\nu_{\underline{x}}$ to the radius of the smallest ball that contains $\eta^\nu_{\underline{x}}$. A similar condition is also assumed in the classical FEM. If this ratio is uniformly bounded for all $\underline{x} \in A^\nu_\Omega$, for all ν, then (6.13) holds.

Remark 6.5 In practical computations, one can easily construct particle-shape function systems (with $k = 0$), such that conditions (6.2), (6.3), (6.13), and conditions (a), (b) of Theorem 6.3 are satisfied.

Remark 6.6 We observed that a partition of unity is the staring point for the construction of approximating space for GFEM. It is important to emphasize that construction of partition unity for $k = 0$ is simple, *e.g.,* it could be constructed by Shepard's approach as discussed in Section 4.

Remark 6.7 We have assumed that our particle-shape function system satisfies A1-A7 with $k = 0$ (*i.e.*, it reproduces polynomials of degree 0), and have seen that the quality of the approximation in Theorems 6.1–6.3 depends entirely on the approximability properties of the spaces $V^\nu_{\underline{x}}$, as quantified by $\epsilon_1(\underline{x})$ and $\epsilon_2(\underline{x})$. If we used a particle-shape function system that reproduced polynomials of degree 1 ($k = 1$), then the space V^ν defined in 6.6 would be enlarged, and its approximability would be improved, possibility only marginally, but this improvement would not be directly visible from (6.11) (or (6.14) or (6.18)). Note that Theorems 6.1–6.3 are directed toward the use of nonpolynomial approximating functions, where the rate of convergence cannot be easily defined.

To clarify this point, suppose for $\phi^\nu_{\underline{x}}$ we use the usual FE hat functions of degree 1, and $V^\nu_{\underline{x}}$ is the space of constants. Then the GFEM is the classical

FEM, with the usual rate of convergence of $O(h)$. However, (6.11) (or (6.14) or (6.18)) does not establish this rate. As a second example, let $V_{\underline{x}}^{\nu}$ be the space of linear polynomials. Then the GFEM is a FE method, but not a usual one. The method has the rate of convergence $O(h^2)$, but (6.11) (or (6.14) or (6.18)) only establishes $O(h)$.

More generally, if $\{\phi_{\underline{x}}^{\nu}\}$ reproduces polynomials of degree r, then the functions $\phi_{\underline{x}}^{\nu}\psi_{\underline{x}}^{\nu}$, which are used in V^{ν}, reproduce polynomials on degree $k+r$. This observation allows one too establish the higher rate of convergence noted in the previous paragraph. This will be done in a forthcoming paper.

The estimates in Theorems 6.2 and 6.3 are quite general, and allow us to employ available information on the approximated function u. Convergence of the approximation can be obtained by considering $\nu_i \in N,\ i = 1, 2, \dots,$ such that $h^{\nu_i} \searrow 0$, where h^{ν} is defined in (3.80). This is reminiscent of the h-version of FEM. Convergence of the approximation can also be attained by keeping ν fixed, and selecting a sequence of spaces $V_{\underline{x}}^{\nu,i},\ i = 1, 2, \dots,$ so that they are complete in $H^1(\mathring{\eta}_{\underline{x}}^{\nu})$ or in a space $\mathcal{W}(\mathring{\eta}_{\underline{x}}^{\nu}) \subset H^1(\mathring{\eta}_{\underline{x}}^{\nu})$ that is known to include the approximated function u_0. This is a generalization of the p-version of FEM.

6.2 Selection of $V_{\bar{x}}$ and "Handbook" Problems

We saw in Section 6.1 that it is important to select spaces $V_{\underline{x}}^{\nu}$ with good local approximation properties. Principles for selecting shape functions that take advantage of available information on the approximated function were formulated in [14, 12]. We will use these ideas to discuss the selection of the space $V_{\underline{x}}^{\nu}$. In this section we will suppress ν in our notation.

Let $H_1(\eta_{\underline{x}})$ and $H_2(\eta_{\underline{x}})$ be two Hilbert spaces, and suppose $H_2(\eta_{\underline{x}}) \subset H_1(\eta_{\underline{x}})$. Then

$$d_n(H_2, H_1) = \inf_{\substack{S_n \subset H_1 \\ \dim S_n = n}} \ \sup_{\substack{u \in H_2 \\ \|u\|_{H_2} \leq 1}} \ \inf_{\chi \in S_n} \|u - \chi\|_{H_1}$$

is called the n-width of H_2-unit ball in H_1. Let $V_{\underline{x}}^{(n)}$ be an n-dimensional subspace of H_1, and let

$$\Psi(V_{\underline{x}}^{(n)}, H_2, H_1) = \sup_{\substack{u \in H_2 \\ \|u\|_{H_2} \leq 1}} \ \inf_{\chi \in V_{\underline{x}}^{(n)}} \|u - \chi\|_{H_1}.$$

$\Psi(V_{\underline{x}}^{(n)}, H_2, H_1)$ is called the *sup-inf*. We will write $\Psi(V_{\underline{x}}^{(n)})$ for $\Psi(V_{\underline{x}}^{(n)}, H_2, H_1)$ if there is no confusion about the spaces H_1 and H_2. It is clear that

$$d_n(H_2, H_1) = \inf_{\substack{V_{\underline{x}}^{(n)} \subset H_1 \\ \dim V_{\underline{x}}^{(n)} = n}} \Psi(V_{\underline{x}}^{(n)}, H_2, H_1).$$

If an n-dimensional subspace ${}^0V_{\underline{x}}^{(n)}$ satisfies

$$\Psi({}^0V_{\underline{x}}^{(n)}, H_2, H_1) \leq C d_n(H_2, H_1),$$

where $C > 1$ is a constant, independent of n, then we will refer to ${}^0V_{\underline{x}}^{(n)}$ as a nearly optimal subspace relative to H_1 and H_2. An n-dimensional subspace ${}^0\bar{V}_{\underline{x}}^{(n)}$ that satisfies

$$\Psi({}^0\bar{V}_{\underline{x}}^{(n)}, H_2, H_1) = d_n(H_2, H_1),$$

is referred to as optimal subspace relative to H_1 and H_2. An optimal subspace ${}^0\bar{V}_{\underline{x}}^{(n)}$ leads to the minimal error that can be achieved with an n-dimensional space, namely, $d_n(H_2, H_1)$; a nearly optimal subspace leads to essentially the same error, $d_n(H_2, H_1)$.

Suppose we are interested in using the GFEM to approximate the solution u_0 of the Dirichlet problem,

$$\begin{cases} \triangle u_0 = 0 & \text{in } \Omega, \\ u_0 = g, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$. Then, for each $\underline{x} \in X^\nu$, we seek a finite dimensional space $V_{\underline{x}}$ that contains a good approximation $\psi_{\underline{x}}$ to u_0 on $\eta_{\underline{x}}$ (*cf.* (6.7), (6.8)). This will be done by taking advantage of the available information on $u_0\big|_{\eta_{\underline{x}}\cap\Omega}$, namely that $u_0\big|_{\eta_{\underline{x}}\cap\Omega}$ is harmonic. We now illustrate this procedure.

We now suppose that $\eta_{\underline{x}}$ is a disk in $\mathbb{R}^2$ and, for the sake of simplicity, suppose $\eta_{\underline{x}}$ is the unit disk. Let $H_1 = \{u \in H^1(\mathring{\eta}_{\underline{x}}) : u \text{ is harmonic in } \eta_{\underline{x}}\}$. For the space H_2, we use $\mathcal{W}(\mathring{\eta}_{\underline{x}})$, a (regularity) space known to contain u_0. More precisely, we suppose $\mathcal{W}(\mathring{\eta}_{\underline{x}})$ is a linear manifold in $\{u \in H^1(\mathring{\eta}_{\underline{x}}) : u \text{ is harmonic}\}$ and suppose $|||u|||$ is a norm on $\mathcal{W}(\mathring{\eta}_{\underline{x}})$ that is rotationally invariant and satisfies $\|u\|_{H^1(\mathring{\eta}_{\underline{x}})} \leq |||u|||$, for all $u \in \mathcal{W}(\mathring{\eta}_{\underline{x}})$. Moreover, we assume $\mathcal{W}(\mathring{\eta}_{\underline{x}})$ is complete with respect to $|||\cdot|||$, *i.e.,* $\{\mathcal{W}(\mathring{\eta}_{\underline{x}}), |||\cdot|||\}$ is a Hilbert space. We note that $\mathcal{W}(\mathring{\eta}_{\underline{x}})$ could be any higher order (isotropic) Sobolev space.

It is well known that any $u \in H_1$ is characterized by its trace on the boundary $I = \partial\eta_{\underline{x}}$; these traces will be in

$$S = \{u(\theta) : 0 < \theta \leq 2\pi, u \text{ is } 2\pi \text{ periodic}, u \in H^{1/2}(I)\}.$$

Any $u \in S$ can be expanded in its Fourier series

$$u(\theta) = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta). \tag{6.27}$$

It is immediate that

$$|u|^2_{H^{1/2}(I)} = a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)k,$$

where $|u|_{H^{1/2}(I)}$ is a Sobolev norm of order 1/2 on I. So we have a one-to-one correspondence between $u(r,\theta) \in H_1$ ((r,θ) are polar coordinates) and $u(\theta) \in S$, which we express by writing $u(r,\theta) \sim u(\theta)$. We easily find that

$$\|u\|^2_{H^1(\mathring{\eta}_{\underline{x}})} = |u|^2_{H^{1/2}(I)} = a_0^2 + \sum_{j=1}^{\infty}(a_j^2 + b_j^2)j. \tag{6.28}$$

Thus we identify the space H_1 with $H^{1/2}(I)$.

Since $|||u|||$ is rotationally invariant, the corresponding norm on $u(\theta)$ will be translation invariant, and we thus can show that

$$|||u|||^2 = a_0^2 + \sum_{j=1}^{\infty}(a_j^2 + b_j^2) j\beta_j, \tag{6.29}$$

where, since $\|u\|_{H^1(\hat{\eta}_{\underline{x}})} \leq |||u|||$, we have $\beta_j \geq 1$. If we now define

$$H^\beta(I) = \{u \in S : |u|_\beta < \infty\},$$

where

$$|u|_\beta^2 = a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) k\beta_k, \tag{6.30}$$

then we see that $u(r,\theta) \in H_2$ if and only if $u(\theta) \in H^\beta(I)$ and $|||u||| = |u|_\beta$. We thus identify the space H_2 with $H^\beta(I)$.

We will now find an optimal subspace ${}^0V_{\underline{x}}^{(n)}$ relative to H_1 and H_2. We will exploit the correspondence $u(r,\theta) \sim u(\theta)$, and find ${}^0V_{\underline{x}}^{(n)}$ by first identifying an optimal subspace relative to $\bar{H}_1 = H^{1/2}(I)$ and $\bar{H}_2 = H^\beta(I)$.

Let $M_n = \{m_1, m_2, \ldots, m_n\}$ be a set of n positive integers, and consider

$$V^{M_n} = \{u \in H^{1/2}(I) : u = a_0 + \sum_{k \in M_n}(a_k \cos k\theta + b_k \sin k\theta)\}. \tag{6.31}$$

Clearly, V^{M_n} is a $(2n+1)$-dimensional space.

Lemma 6.1 *Let $\bar{H}_1 = H^{1/2}(I)$, $\bar{H}_2 = H^\beta(I)$, where $\beta = (\beta_1, \beta_2, \cdots)$, $\beta_k \geq 1$, and let V^{M_n} be as defined in (6.31). Then*

$$\Psi(V^{M_n}, \bar{H}_2, \bar{H}_1) = (\gamma(V^{M_n}))^{-\frac{1}{2}}, \tag{6.32}$$

where

$$\gamma(V^{M_n}) = \inf_{i \notin M_n} \beta_i.$$

Proof. Consider $u \in \bar{H}_2$ given by

$$u = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta).$$

Then from (6.31), we get

$$\inf_{\chi \in V^{M_n}} |u - \chi|_{\bar{H}_1}^2 = \sum_{k \in N - M_n}(a_k^2 + b_k^2)k,$$

where N is the set of all positive integers. Therefore from (6.30) and the definition of $\gamma(V^{M_n})$, we have

$$\begin{aligned}\inf_{\chi\in V^{M_n}} \frac{|u-\chi|^2_{\bar H_1}}{|u|^2_{\bar H_2}} &= \frac{\sum_{k\in N-M_n}(a_k^2+b_k^2)k}{a_0^2+\sum_{k\in N}(a_k^2+b_k^2)k\beta_k}\\ &\le \frac{\sum_{k\in N-M_n}(a_k^2+b_k^2)k}{\sum_{k\in N-M_n}(a_k^2+b_k^2)k\beta_k}\\ &\le \frac{1}{\gamma(V^{M_n})}.\end{aligned}$$

Thus,

$$\sup_{u\in\bar H_2}\inf_{\chi\in V^{M_n}} \frac{|u-\chi|^2_{\bar H_1}}{|u|^2_{\bar H_2}} \le \frac{1}{\gamma(V^{M_n})}. \tag{6.33}$$

Let $\epsilon>0$ be arbitrary. Then there is an $m_0\notin M_n$, $m_0\ge 1$, such that

$$\beta_{m_0}\le \gamma(V^{M_n})+\epsilon. \tag{6.34}$$

Consider $u_{m_0}=\cos m_0\theta$. Clearly, $u_{m_0}\notin V^{M_n}$, and therefore from (6.27),

$$\inf_{\chi\in V^{M_n}}|u_{m_0}-\chi|^2_{\bar H_1}=|u_{m_0}|^2_{\bar H_1}=m_0.$$

Also, from (6.30), we have $|u_{m_0}|^2_{\bar H_2}=m_0\beta_{m_0}$. Therefore, using (6.34), we get

$$\sup_{u\in\bar H_2}\inf_{\chi\in V^{M_n}} \frac{|u-\chi|^2_{\bar H_1}}{|u|^2_{\bar H_2}} \ge \inf_{\chi\in V^{M_n}}\frac{|u_{m_0}-\chi|^2_{\bar H_1}}{|u_{m_0}|^2_{\bar H_2}}=\frac{1}{\beta_{m_0}}\ge\frac{1}{\gamma(V^{M_n})+\epsilon}.$$

From this estimate and (6.33), we have

$$\frac{1}{\gamma(V^{M_n})+\epsilon}\le \sup_{u\in\bar H_2}\inf_{\chi\in V^{M_n}} \frac{|u-\chi|^2_{\bar H_1}}{|u|^2_{\bar H_2}} \le \frac{1}{\gamma(V^{M_n})}.$$

Since ϵ is arbitrary, we get (6.32). □

Lemma 6.2 *Let $\bar H_1=H^{1/2}(I)$ and $\bar H_2=H^{\beta}(I)$, where $\beta=(\beta_1,\beta_2,\cdots)$, $\beta_k\ge 1$. Then*

$$d_{2n}(H_2,H_1)=(\gamma_n^*)^{-\frac12},$$

where

$$\gamma_n^*=\sup_{m_1,m_2,\ldots,m_n}\inf_{i\notin M_n}\beta_i.$$

The proof of this theorem follows immediately from Lemma 6.1.

Theorem 6.4 *Suppose* $H_1 = \{u \in H^1(\mathring{\eta}_{\underline{x}}) : u \text{ is harmonic}\}$ *and* $H_2 = \mathcal{W}(\mathring{\eta}_{\underline{x}})$ *with the norm* $|||u|||_\beta = |u|_\beta$, *given in (6.30), with* $\beta_j \geq 1$. *Suppose in addition that the sequence* β_j *is non-decreasing. Then the space*

$$^0V_{\underline{x}}^{(2n+1)} = span\{r^j \cos j\theta, r^j \sin j\theta\}_{j=0}^n$$

i.e., the span of first $(2n+1)$ *harmonic polynomials is optimal relative to* H_1 *and any* H_2 *(i.e., any of the spaces* H_2 *we are considering).*

Proof. Using the correspondence $u(r, \theta) \sim u(\theta)$, we can study the optimality of a finite dimensional subspace relative to H_1 and H_2, by studying the optimality of a subspace relative to $\bar{H}_1$ and $\bar{H}_2$. The result follows directly from Lemma 6.2. □

Remark 6.8 Obviously the condition on β in Theorem 6.4 holds for any (iotropic) Sobolev space.

Remark 6.9 Let us return to the solution of the Dirichelet problem mentioned above. Suppose $\eta_{\underline{x}}$ is far from the boundary of Ω. Then on $\eta_{\underline{x}}$, the character of the solution u_0 is approximately the same in any direction. Thus it is appropriate to embed u_0 in a space with a rotationally invariant norm—a usual (isotropic) Sobolev space, *e.g.* And we have learned that on $\eta_{\underline{x}}$, u_0 is well approximated by harmonic polynomials. The situation is, however, somewhat different when $\eta_{\underline{x}}$ is near the bounday. Then u_0 would be strongly influenced by the boundary values $g(x)$. Hence some other shape functions, constructed, *e.g.*, by the Handbook approach, which themselves reflected these boundary values, would be "best".

Thus the optimal shape functions are the solution of the Laplace equation. This approach could be also be used for other differential equation, *e.g.*, $-\triangle u + ku = 0$, or when $\eta_{\underline{x}} \cap \Omega = B_1 - B_{1/2}$ where B_ρ is the ball of diameter ρ and homogeneous normal boundary conditions are prescribed on $\partial\Omega$.

In this section, we saw an example of choosing an optimal local approximating space $V_{\underline{x}}$, which turned out to be the span of first $(2n+1)$ harmonic polynomials. In other problems, different local approximating spaces, consisting of optimal or near optimal approximating functions, are recommended. These optimal or near optimal approximating functions are solutions of other boundary value problems (posed on $\eta_{\underline{x}} \cap \Omega$). Such locally posed problems are called *Handbook Problems*, and their solutions, which may be available analytically or computed numerically, are called *Handbook Functions*. This nomenclature is reminiscent of the solved problems and their solutions (via formulae, tables *etc.*) which are used in engineering ([86]). This idea is also used in commercial codes ([86, 85]).

One of the main advantages of GFEM is that only simple meshes are used, which need not reflect the boundary, *e.g.,* uniform finite element meshes. Also, in each $\eta_{\underline{x}}$, one can use a space $V_{\underline{x}}$ of arbitrary dimension (depending on $\underline{x}$). $V_{\underline{x}}$

could be space of polynomials or any other space of functions depending on the local properties of the approximated function.

Choosing $V_{\underline{x}}$ to be the space of polynomials of low degree p (and using $\{\phi_{\underline{x}}\}$ that are reproducing of order k), we obtain the h-version of FEM. All other classical versions of FEM—the p and h-p versions— are special cases of GFEM. Some examples of the use of this method will be discussed in Section 9.

7 Solutions of Elliptic Boundary Value Problem

In this section, we will discuss the approximate solution of the model problem (2.1)–(2.2) (or (2.3)), introduced in Section 2, by a meshless method. We will address the the Neumann boundary condition (2.2) and the Dirichlet boundary condition (2.3) separately. These problems have the variational formulation (2.4).

For $0 < h \leq 1$, we consider a family of particle-shape function systems

$$\{\mathcal{M}^h\}_{0<h\leq 1} = \{X^h, \{h^h_{\underline{x}},\, \omega^h_{\underline{x}},\, \phi^h_{\underline{x}}\}_{\underline{x}\in X^h}\}_{0<h\leq 1},$$

satisfying the assumptions A1–A7 in Section 3.3 and (3.64). Recall that (3.64) is trivially satisfied if the shape functions are reproducing of order k. The family $\{\mathcal{M}^h\}_{0<h\leq 1}$ was introduced in Section 3.3; recall that

$$\sup_{\underline{x}\in X^h} h^h_{\underline{x}} \leq h.$$

We will be interested in assessing the approximation error as $h \searrow 0$.

Let u_0 be the solution of (2.4), where Ω is a bounded domain with Lipschitz continuous boundary. Sometimes in this section, we will assume that the boundary of Ω is smooth. We will use the space $\mathbb{V}^{k,q}_{\Omega,h}$, defined in (3.86), to approximate u_0. It was shown in Theorem 3.11 that $\mathbb{V}^{k,q}_{\Omega,h}$ is $(k+1,q)$ regular. Moreover, $\mathbb{V}^{k,q}_{\Omega,h}$ satisfies the local assumption LA. Recall that k is the order of the quasi-reproducing shape functions considered in $\{\mathcal{M}^h\}$ and q is the smoothness index of these shape functions. The parameters k and q are in the assumptions A1–A7 and we assume that $q \leq k+1$. We also recall that $\mathbb{V}^{k,q}_{\Omega,h}$ does not involve all the particles in X^h. It only involves particles in the set

$$A^h_\Omega = \{\underline{x} \in X^h : \eta^h_{\underline{x}} \cap \Omega \neq \emptyset\}. \tag{7.1}$$

Various classes of shape functions can be used for $\phi^h_{\underline{x}}$ in the system $\{\mathcal{M}^h\}$. In Section 4, one such class of shape functions, namely RKP shape functions, were discussed, and references related to other classes of shape functions used in practice were provided.

We note that it is possible to construct particle-shape function system $\mathcal{M}^h$, satisfying A1–A7, such that the set of particles $X^h \subset \Omega$, and the corresponding $\mathbb{V}^{k,q}_{\Omega,h}$ have desired approximation properties. We do not consider such $\mathbb{V}^{k,q}_{\Omega,h}$ in this section, and we will further remark on this issue in the next subsection.

Let $u_S = u_h \in \mathbb{V}^{k,q}_{\Omega,h}$ be the approximate solution defined by (2.7) with $S = \mathbb{V}^{k,q}_{\Omega,h}$. Since $\mathbb{V}^{k,q}_{\Omega,h}$ is $(k+1,q)$-regular, we note that $\mathbb{V}^{k,q}_{\Omega,h} \subset H = H^1(\Omega)$ provided $q \geq 1$. Thus u_h is the solution of

$$\begin{cases} u_h \in \mathbb{V}^{k,q}_{\Omega,h} \\ \tilde{B}(u_0, v) = \displaystyle\int_\Omega f v \, dx, \quad \text{for all } v \in \mathbb{V}^{k,q}_{\Omega,h}, \end{cases} \tag{7.2}$$

where the bilinear form $\tilde{B}$ is either B, given in (2.5), or a perturbation of B. Clearly, u_h is the solution of a Galerkin method. This Galerkin method is a *meshless method* since the construction of the test and the trial space, *i.e.,* $\mathbb{V}^{k,q}_{\Omega,h}$, does not require a mesh. As we remarked in Section 1, avoiding mesh generation is one of the main features and advantages of meshless methods.

In this section, we will consider u_h as an approximation of u_0 and primarily study the error $u_0 - u_h$. We set some notations that will be used in this study in the following sections. We define

$$E^h_{\underline{x}} \equiv \mathring{\eta}^h_{\underline{x}} \cap \partial\Omega, \quad \underline{x} \in A^h_\Omega, \tag{7.3}$$

and

$$A^h_{\partial\Omega} = \{\underline{x} \in A_\Omega : E^h_{\underline{x}} \neq \emptyset\}. \tag{7.4}$$

Thus $A^h_{\partial\Omega}$ is the set of particles $\{\underline{x}\}$ such that $\mathring{\eta}^h_{\underline{x}}$ has non-empty intersection with $\partial\Omega$.

7.1 A Meshless Method for Neumann Boundary Condition

In this section, we will address the approximation of solution u_0 of (2.1) and (2.2) by the meshless method. The analysis presented here is based on the ideas and results in [5, 11]. Also see the references listed in these articles.

The solution u_0 of (2.1), (2.2) can be variationally characterized by (2.4), which is

$$\begin{cases} u_0 \in H^1(\Omega) \\ B(u_0, v) = \displaystyle\int_\Omega f v \, dx, \quad \text{for all } v \in H^1(\Omega). \end{cases} \tag{7.5}$$

We wish to approximate u_0 by u_h, the solution of (7.2) with $\tilde{B} = B$. For an error estimate, from (2.8) we have

$$\|u_0 - u_h\|_{H^1(\Omega)} \leq \inf_{\chi \in \mathbb{V}^{k,q}_{\Omega,h}} \|u_0 - \chi\|_{H^1(\Omega)}.$$

Suppose $u_0 \in H^l(\Omega)$. Then, since $\mathbb{V}^{k,q}_{\Omega,h}$ is $(k+1,q)$-regular and $q \geq 1$, we have

$$\|u_0 - u_h\|_{H^1(\Omega)} \leq C h^\mu \|u_0\|_{H^l(\Omega)},$$

where $\mu = \min(k, l-1)$. We summarize this in the following theorem.

Theorem 7.1 *Suppose $u_0 \in H^l(\Omega)$, with $l \geq 1$, is the solution of (7.5), where $\partial\Omega$ is Lipschitz. Let $u_h \in \mathbb{V}^{k,q}_{\Omega,h}$, with $q \geq 1$, be the approximate solution given by (7.2) with $\tilde{B} = B$. Then*

$$\|u_0 - u_h\|_{H^1(\Omega)} \leq h^{\mu}\|u_0\|_{H^l(\Omega)}, \tag{7.6}$$

where

$$\mu = \min(k, l-1) \tag{7.7}$$

We note that the computation of u_h, in Theorem 7.1, depends on the definition of $\mathbb{V}^{k,q}_{\Omega,h}$ and involves particles that are also outside Ω. In the literature, especially in the engineering literature, (t,k^*)-regular particle spaces are constructed using particles inside Ω, but the support of some of the corresponding particle shape functions could be partly outside Ω. The apparent reason for such construction is that the approximate solution is viewed as an interpolant with respect to data inside Ω, and hence the particles that are only inside Ω are considered. This certainly is not necessary.

The construction of the approximation space S (in (2.7)) using particles only inside Ω sometimes may lead to better conditioning of the underlying linear system. On the other hand, such construction is more expensive and the approximations could show boundary layer behavior ([13]).

7.2 Meshless Methods for Dirichlet Boundary Condition

In this section, we consider the approximation of the solution u_0 of the Dirichlet boundary value problem (2.1) and (2.3) by meshless methods. The variational characterization of u_0 is given by

$$\left\{\begin{array}{l} u_0 \in H^1_0(\Omega) \\ B(u_0, v) = \displaystyle\int_\Omega f v\, dx, \quad \text{for all } v \in H^1_0(\Omega). \end{array}\right. \tag{7.8}$$

The Galerkin method (2.7) to approximate u_0 would require that the approximating space S be a subspace of $H = H^1_0(\Omega)$ and thus that the approximating functions satisfy the essential homogeneous Dirichlet boundary condition. Unlike shape functions used in FEM, the particle shape functions $\phi^h_{\underline{x}}$ (we consider h as the parameter), considered in Section 3.3, do not in general satisfy the so called "Kronecker delta" property, *i.e.*, $\phi^h_{\underline{x}}(\underline{y}) \neq \delta_{\underline{x},\underline{y}}$, $\underline{x}, \underline{y} \in X^h$. This is also true for translation invariant particle shape functions discussed in Section 3.2 (see Section 4.2). Thus it is difficult to construct a subspace $S \subset \mathbb{V}^{k,q}_{\Omega,h}$ such that S could be used in (2.7) as the approximation space and the functions in S satisfy the Dirichlet boundary condition.

In the literature, several meshless methods have been proposed to approximate the solutions of Dirichlet boundary value problems. They are meshless methods in the sense that they use $\mathbb{V}^{k,q}_{\Omega,h}$ as the approximating space. These methods are:

1. The Penalty Method
2. The Lagrange Multiplier Method
3. The Nitsche and Related Methods
4. The Collocation Method
5. Combination of meshless and finite element method
6. The characteristic function method

In this section we will describe these methods. We note that GFEM, discussed in Section 6, uses an approximating space, different from $\mathbb{V}_{\Omega,h}^{k,q}$, and can also be used to approximate the solution of a Dirichlet boundary value problem.

We will assume that the boundary $\partial\Omega$ of Ω is sufficiently smooth. The smoothness assumption on the boundary simplifies the arguments presented here, but various results could be obtained when the boundary is not smooth.

1. The Penalty Method.

The main idea of the penalty method is to use a perturbed variational principle. For $\sigma > 0$, we consider the bilinear form

$$\tilde{B}(u,v) \equiv B_\sigma(u,v) \equiv B(u,v) + h^{-\sigma} D(u,v), \tag{7.9}$$

where

$$B(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx, \tag{7.10}$$

$$D(u,v) = \int_{\partial\Omega} uv\, dx. \tag{7.11}$$

We note that (7.10) is the bilinear form given in (2.5). We consider the solution $u_h = u_{\sigma,h} \in \mathbb{V}_{\Omega,h}^{k,q}$ of (7.2), namely

$$B_\sigma(u_{\sigma,h}, v) = \int_\Omega fv\, dx, \quad \text{for all } v \in \mathbb{V}_{\Omega,h}^{k,q}. \tag{7.12}$$

We note that $u_{\sigma,h}$ is u_S, where u_S is defined in (2.7). For $v \in H^1(\Omega)$, let

$$Q_\sigma(v) = B(v,v) + h^{-\sigma} D(v,v) - 2\int_\Omega fv\, dx. \tag{7.13}$$

It is well known that

$$Q_\sigma(u_{\sigma,h}) = \min_{v \in \mathbb{V}_{\Omega,h}^{k,q}} Q_\sigma(v). \tag{7.14}$$

We now present a convergence result for the penalty method.

Theorem 7.2 *Suppose $u_0 \in H^l(\Omega) \cap H_0^1(\Omega)$, $l > 3/2$, is the solution of (7.8). Let $u_{\sigma,h} \in \mathbb{V}_{\Omega,h}^{k,q}$ be the solution of (7.12). Then for any $0 < \epsilon < \min(l - 3/2,\ 1/2)$, we have*

$$\|u_0 - u_{\sigma,h}\|_{H^1(\Omega)} \le C(\epsilon) h^\mu \|u_0\|_{H^l(\Omega)}, \tag{7.15}$$

where

$$\mu = \min(k,\, l-1,\, \frac{\sigma}{2},\, k+\frac{1}{2}-\frac{\sigma}{2}-\epsilon,\, l-\frac{1}{2}-\frac{\sigma}{2}-\epsilon), \tag{7.16}$$

and $C(\epsilon)$ depends on ϵ, but is independent of h and u_0.

Proof. For any $v \in H^1(\Omega)$, we define

$$R_\sigma(v) = B(u_0 - v, u_0 - v) + h^{-\sigma} D(\frac{\partial u_0}{\partial n} h^\sigma + v, \frac{\partial u_0}{\partial n} h^\sigma + v). \tag{7.17}$$

Then from Green's Theorem,

$$\begin{aligned} R_\sigma(v) &= B(u_0, u_0) + B(v, v) - 2B(u_0, v) \\ &\qquad + h^\sigma D(\frac{\partial u_0}{\partial n}, \frac{\partial u_0}{\partial n}) + h^{-\sigma} D(v, v) + 2D(\frac{\partial u_0}{\partial n}, v) \\ &= B(u_0, u_0) + h^\sigma D(\frac{\partial u_0}{\partial n}, \frac{\partial u_0}{\partial n}) \\ &\qquad + B(v, v) + h^{-\sigma} D(v, v) - 2\int_\Omega f v\, dx \\ &= B(u_0, u_0) + h^\sigma D(\frac{\partial u_0}{\partial n}, \frac{\partial u_0}{\partial n}) + Q_\sigma(v), \quad \text{for all } v \in H^1(\Omega), \end{aligned}$$

where $Q_\sigma(v)$ is given by (7.13). Therefore,

$$\min_{v \in \mathbb{V}^{k,q}_{\Omega,h}} R_\sigma(v) = B(u_0, u_0) + h^\sigma D(\frac{\partial u_0}{\partial n}, \frac{\partial u_0}{\partial n}) + \min_{v \in \mathbb{V}^{k,q}_{\Omega,h}} Q_\sigma(v),$$

and thus from (7.14) we get,

$$R_\sigma(u_{\sigma,h}) = \min_{v \in \mathbb{V}^{k,q}_{\Omega,h}} R_\sigma(v).$$

Hence, from (7.17) and the above relation, we have

$$\begin{aligned} \|u_0 - u_{\sigma,h}\|^2_{H^1(\Omega)} &= B(u_0 - u_{\sigma,h}, u_0 - u_{\sigma,h}) \\ &\le R_\sigma(u_{\sigma,h}) \\ &\le R_\sigma(v), \quad \text{for all } v \in \mathbb{V}^{k,q}_{\Omega,h}. \end{aligned} \tag{7.18}$$

Since $\mathbb{V}^{k,q}_{\Omega,h}$ is $(k+1, q)$-regular with $q \ge 1$, there is $g_h \in \mathbb{V}^{k,q}_{\Omega,h}$ such that

$$\|u_0 - g_h\|_{H^s(\Omega)} \le C h^\mu \|u_0\|_{H^l(\Omega)}, \tag{7.19}$$

where $\mu = \min(k+1-s, l-s)$ and $0 \le s \le 1$. Now from (7.17) with $v = g_h$ and using Cauchy-Schwartz inequality we have

$$R_\sigma(g_h) \le C\left(\|u_0 - g_h\|^2_{H^1(\Omega)} + h^\sigma \int_{\partial\Omega} (\frac{\partial u_0}{\partial n})^2\, ds + h^{-\sigma} \int_{\partial\Omega} g_h^2\, ds\right). \tag{7.20}$$

We will estimate the right hand side of the above inequality. We first note that $u_0 = 0$ on $\partial\Omega$. Let $0 < \epsilon < \min(l - \frac{3}{2}, \frac{1}{2})$. Then using a trace inequality and (7.19) with $s = (1/2) + \epsilon$, we get

$$\begin{aligned} \|g_h\|^2_{L_2(\partial\Omega)} = \|u_0 - g_h\|^2_{L_2(\partial\Omega)} &\leq C(\epsilon)\|u_0 - g_h\|^2_{H^{\frac{1}{2}+\epsilon}(\Omega)} \\ &\leq C(\epsilon)h^{2\mu_1}\|u_0\|^2_{H^l(\Omega)}, \end{aligned} \tag{7.21}$$

where $\mu_1 = \min(k + \frac{1}{2} - \epsilon,\ l - \frac{1}{2} - \epsilon)$. Also from a trace inequality, we have

$$\|\frac{\partial u_0}{\partial n}\|^2_{L_2(\partial\Omega)} \leq C(\epsilon)\|u_0\|^2_{H^{\frac{3}{2}+\epsilon}(\Omega)}. \tag{7.22}$$

Now using (7.21), (7.22), and (7.19), with $s = 1$, in (7.20), we get

$$\begin{aligned} R_\sigma(g_h) &\leq C(\epsilon)\left(h^{2\min(k,l-1)} + h^\sigma + h^{2\mu_1-\sigma}\right)\|u_0\|^2_{H^l(\Omega)} \\ &\leq C(\epsilon)h^{2\mu}\|u_0\|^2_{H^l(\Omega)}, \end{aligned} \tag{7.23}$$

where $\mu = \min(k, l-1, \frac{\sigma}{2}, k + \frac{1}{2} - \frac{\sigma}{2} - \epsilon, l - \frac{1}{2} - \frac{\sigma}{2} - \epsilon)$. Finally, combining (7.18) and (7.23) we get the desired result. □

Remark 7.1 If we consider $\mathbb{V}^{k,q}_{\Omega,h}$ in Theorem 7.2 such that $k + 1 \geq l > 3/2$, then with $\sigma = l - \frac{1}{2} - \epsilon$, it is easy to see that (7.15) holds with

$$\mu = \frac{1}{2}(l - \frac{1}{2} - \epsilon).$$

The estimate (7.15) can be improved. We present the following result, based on the analysis in [4, 5, 6, 8], without proof.

Theorem 7.3 *Suppose $u_0 \in H^l(\Omega) \cap H^1_0(\Omega)$ is the solution of (7.8). Let $u_{\alpha,h} \in \mathbb{V}^{k,q}_{\Omega,h}$ be the solution of (7.12). If $k + 1 \geq l \geq 2$, then, for any $\epsilon > 0$, we have*

$$\|u_0 - u_{\alpha,h}\|_{H^1(\Omega)} \leq C(\epsilon)h^{\mu-\epsilon}\|u_0\|_{H^l(\Omega)}, \tag{7.24}$$

where $C(\epsilon)$ is independent of u_0 and h but depends on ϵ, and μ is given by

$$\mu = \min\left(\sigma,\ l + \sigma - 2,\ l + \frac{\sigma}{2} - \frac{3}{2},\ \frac{k + 1 - \kappa}{k}(l - 1)\right), \tag{7.25}$$

where

$$\kappa = \max(1, \frac{1+\sigma}{2}). \tag{7.26}$$

Remark 7.2 If $\sigma \geq 1$ in Theorem 7.3, then $l + \sigma - 2 \geq l + \frac{\sigma}{2} - \frac{3}{2}$, and therefore μ in (7.24) is given by

$$\mu = \min\left(\sigma, l + \frac{\sigma}{2} - \frac{3}{2}, \frac{k + 1 - \kappa}{k}(l - 1)\right), \tag{7.27}$$

where κ is as in (7.26).

Example: Consider $l = 2$ and $\sigma = 1$ in Theorem 7.3. Then from (7.26), we have $\kappa = 1$, and (7.27) yields $\mu = \min(1, 1, 1) = 1$. This is the optimal rate of convergence. For higher values of l and $k + 1 \geq l$, there is a loss in the rate of convergence and we get sub-optimal rate of convergence, *i.e.,* $\mu < \min(l - 1, k)$.

Thus from Theorem 7.3, we conclude the following:

- It is advantageous to use $\mathbb{V}_{\Omega,h}^{k,q}$ with higher values of k since it leads to higher accuracy. For example, if $l = 4$ and $\sigma = 3$, then $\kappa = 2$ and (7.27) yields $\mu = 3(\frac{k-1}{k})$. Thus higher values of k will increase accuracy. But higher values of k reduce the sparsity of the resulting linear system.

- Too small or too large value of σ may decrease the accuracy significantly. For example if $\sigma = 2k + 1$, then $\kappa = k + 1$ and 7.27) yields $\mu = 0$.

The use of penalty method was recently suggested in the literatue, *e.g.,* [3], [57], [89], without any theoretical analysis. A concrete penalty value of σ, unrelated to l or k, was suggested in [89] based on experience.

2. The Lagrange Multiplier Method

The theory of Lagrange multiplier method, in the context of finite element method, was developed in [7] (see also [11]). This theory can also be extended to meshless methods.

It is known (*cf.* [11]) that the effectiveness of this method depends on a delicate relation between the approximating space $S_h^{t,k^*}(\Omega)$ and the space of Lagrange multipliers $S_{h_1}^{t,k^*}(\partial\Omega)$, where both $S_h^{t,k^*}(\Omega)$ and $S_{h_1}^{t,k^*}(\partial\Omega)$ satisfy an inverse assumption. In the context of meshless methods, we let the approximating space to be the particle space $\mathbb{V}_{\Omega,h}^{k,q}$. We know that $\mathbb{V}_{\Omega,h}^{k,q}$ is $(k+1, q)$-regular, and satisfies the inverse assumption, IA, under additional hypotheses given at the end of Section 3.3. The space of Lagrange multipliers $S_{h_1}^{k+1,q}(\partial\Omega)$ has the same (t, k^*)-regularity as the approximating space, and the functions in $S_{h_1}^{k+1,q}(\partial\Omega)$ are defined only on $\partial\Omega$ with respect to particles on $\partial\Omega$. Thus, $\partial\Omega$ must contain enough particles. We note that the functions in $S_{h_1}^{k+1,q}(\partial\Omega)$ are not restrictions of functions in $\mathbb{V}_{\Omega,h}^{k,q}$ on $\partial\Omega$. Then following the analysis in [7, 11], one can show that if the size of the supports of the basis elements of $S_{h_1}^{k+1,q}(\partial\Omega)$ is of the same order as $|E_{\underline{x}}^h|$, $\underline{x} \in A_{\partial\Omega}^h$ ($E_{\underline{x}}^h$ and $A_{\partial\Omega}^h$ defined in (7.3) and (7.4), respectively), then the approximate solution obtained from the Lagrange multiplier method converges. If the size of the supports of the basis elements of $S_{h_1}^{k+1,q}(\partial\Omega)$ is smaller than $\eta_{\underline{x}}^h$, $\underline{x} \in A_{\partial\Omega}^h$, then the method is unstable. This relationship was further analyzed in [72] and [73].

The Lagrange multiplier technique leads to the optimal rate of convergence in comparison to the penalty method, where the optimal rate of convergence is usually not attained. But, as we mentioned before, the sufficient conditions for convergence are quite delicate.

Recently, the Lagrange multiplier technique was applied in the context of meshless methods without any theoretical analysis in [24], [57], [60], [67].

3. The Nitsche and Related Methods

Because of the delicate nature of Lagrange multiplier method, there was an interest to look for other methods to deal with the issue of the imposition of Dirichlet boundary conditions and to avoid complications that are present in Lagrange multiplier method. Towards this end, certain methods were proposed in [20] and [79]. But much earlier, a similar method was proposed by Nitsche in [70]. We will discuss the Nitsche method, following the presentation in [79]. We will still assume that $\partial\Omega$ is smooth.

To approximate the solution u_0 of (7.8) by Nitche method, we consider the particle space $\mathbb{V}^{k,q}_{\Omega,h}$ with $q \geq 2$. We also assume that

- Card $(A^h_{\partial\Omega}) \leq \kappa$ and

$$C_1 h \leq h_{E^h_{\underline{x}}} \leq C_2 h, \quad \underline{x} \in A^h_{\partial\Omega}, \tag{7.28}$$

  where $h_{E^h_{\underline{x}}} \equiv |E^h_{\underline{x}}|$, for $\underline{x} \in A^h_{\partial\Omega}$; $E^h_{\underline{x}}$ and $A^h_{\partial\Omega}$ are defined in (7.3) and (7.4), respectively.

- There is $0 < \mathcal{K} < \infty$, $\mathcal{K} = \mathcal{K}(X^h)$, such that

$$\|\frac{\partial v}{\partial n}\|_{-\frac{1}{2},h} \leq \mathcal{K}[B(v,v)]^{1/2}, \text{ for all } v \in \mathbb{V}^{k,q}_{\Omega,h}, \tag{7.29}$$

  where $B(u,v)$ was defined in (7.10) and

$$\|\frac{\partial v}{\partial n}\|^2_{-\frac{1}{2},h} = \sum_{\underline{x}\in A^h_{\partial\Omega}} h_{E^h_{\underline{x}}} \|\frac{\partial v}{\partial n}\|^2_{H^0(E^h_{\underline{x}})}. \tag{7.30}$$

We define the bilinear form $\tilde{B}(u,v) = B_\gamma(u,v)$, where

$$B_\gamma(u,v) = B(u,v) - D(\frac{\partial u}{\partial n}, v) - D(\frac{\partial v}{\partial n}, u) + \gamma \sum_{\underline{x}\in A^h_{\partial\Omega}} {h_{E^h_{\underline{x}}}}^{-1} \int_{E^h_{\underline{x}}} uv\, ds,$$

with $\gamma > 0$; $B(u,v)$, $D(u,v)$ are as defined in (7.10), (7.11), respectively. The approximate solution $u_{h,\gamma} \in \mathbb{V}^{k,q}_{\Omega,h}$, obtained from the Nitche method, is given by

$$B_\gamma(u_{h,\gamma}, v) = \int_\Omega fv\, dx, \quad \text{for all } v \in \mathbb{V}^{k,q}_{\Omega,h}. \tag{7.31}$$

For $u \in H^2(\Omega)$, we define the norm

$$|||u|||^2 = B(u,u) + \|\frac{\partial u}{\partial n}\|^2_{-\frac{1}{2},h} + \|u\|^2_{\frac{1}{2},h},$$

where

$$\|u\|^2_{\frac{1}{2},h} \quad = \quad \sum_{\underline{x}\in A^h_{\partial\Omega}} {h_{E^h_{\underline{x}}}}^{-1} \|u\|^2_{H^0(E^h_{\underline{x}})},$$

and $\|\frac{\partial u}{\partial n}\|^2_{-\frac{1}{2},h}$ is given by (7.30) with v replaced by u. We first note that from Schwartz inequality, we have

$$\begin{aligned}\sum_{\underline{x}\in A^h_{\partial\Omega}} h^{-1}_{E^h_{\underline{x}}}\int_{E^h_{\underline{x}}} uv\,ds &\leq \sum_{\underline{x}\in A^h_{\partial\Omega}}\left(\int_{E^h_{\underline{x}}} h^{-1}_{E^h_{\underline{x}}}u^2\,ds\right)^{1/2}\left(\int_{E^h_{\underline{x}}} h^{-1}_{E^h_{\underline{x}}}v^2\,ds\right)^{1/2}\\ &\leq [\sum_{\underline{x}\in A^h_{\partial\Omega}} h^{-1}_{E^h_{\underline{x}}}\|u\|^2_{H^0(E^h_{\underline{x}})}]^{1/2}[\sum_{\underline{x}\in A^h_{\partial\Omega}} h^{-1}_{E^h_{\underline{x}}}\|v\|^2_{H^0(E^h_{\underline{x}})}]^{1/2}\\ &= \|u\|_{\frac{1}{2},h}\|v\|_{\frac{1}{2},h}. \end{aligned}\tag{7.32}$$

Also,

$$\begin{aligned}D(u,\frac{\partial v}{\partial n}) &\leq \sum_{\underline{x}\in A^h_{\partial\Omega}}\int_{E^h_{\underline{x}}}|h^{-1/2}_{E^h_{\underline{x}}}u\,h^{1/2}_{E^h_{\underline{x}}}\frac{\partial v}{\partial n}|\,ds\\ &\leq \sum_{\underline{x}\in A^h_{\partial\Omega}} h^{-1/2}_{E^h_{\underline{x}}}\|u\|_{H^0(E^h_{\underline{x}})}h^{1/2}_{E^h_{\underline{x}}}\|\frac{\partial v}{\partial n}\|_{H^0(E^h_{\underline{x}})}\\ &\leq \|u\|_{\frac{1}{2},h}\|\frac{\partial v}{\partial n}\|_{-\frac{1}{2},h}. \end{aligned}\tag{7.33}$$

Using the same arguments used to obtain (7.33), we get

$$D(\frac{\partial u}{\partial n},v)\leq\|\frac{\partial u}{\partial n}\|_{-\frac{1}{2},h}\,\|v\|_{\frac{1}{2},h}. \tag{7.34}$$

Now using (7.32)–(7.34), it is can be easily shown that

$$B_\gamma(u,v)\leq(1+\gamma)\,|||u|||\,|||v|||. \tag{7.35}$$

We now show that for a proper value of γ, the bilinear form $B_\gamma(u,v)$ is coercive.

Lemma 7.1 *Suppose $\mathcal{K}^2<\gamma$, where $\mathcal{K}$ satisfies (7.29). Then,*

$$B_\gamma(v,v)\geq C_1^*|||v|||^2,\quad \textit{for all } v\in\mathbb{V}^{k,q}_{\Omega,h}, \tag{7.36}$$

where $C^*=C^*(X^h)>0$.

Proof. Let $v\in\mathbb{V}^{k,q}_{\Omega,h}$ and let $\epsilon>0$ arbitrary. From the definition of $B_\gamma(u,v)$ and (7.33) with $u=v$, we have,

$$\begin{aligned}B_\gamma(v,v) &= B(v,v)-2D(v,\frac{\partial v}{\partial n})+\gamma\|v\|^2_{\frac{1}{2},h}\\ &\geq B(v,v)-2\|v\|_{\frac{1}{2},h}\|\frac{\partial v}{\partial n}\|_{-\frac{1}{2},h}+\gamma\|v\|^2_{\frac{1}{2},h}\\ &\geq B(v,v)-\epsilon\|v\|^2_{\frac{1}{2},h}-\frac{1}{\epsilon}\|\frac{\partial v}{\partial n}\|^2_{-\frac{1}{2},h}+\gamma\|v\|^2_{\frac{1}{2},h}\\ &= B(v,v)-\frac{1}{\epsilon}\|\frac{\partial v}{\partial n}\|^2_{-\frac{1}{2},h}+(\gamma-\epsilon)\|v\|^2_{\frac{1}{2},h}\\ &\geq (1-\frac{\mathcal{K}^2}{\epsilon})B(v,v)+(\gamma-\epsilon)\|v\|^2_{\frac{1}{2},h}. \end{aligned}$$

Therefore, considering $\epsilon = \frac{1}{2}(\mathcal{K}^2 + \gamma)$ in the above inequality, we get

$$B_\gamma(v, v) \geq C_1[B(v, v) + \|v\|^2_{\frac{1}{2},h}], \tag{7.37}$$

where $C_1 = \min(\frac{\gamma-\mathcal{K}^2}{\gamma+\mathcal{K}^2}, \frac{\gamma-\mathcal{K}^2}{2})$. Now, from the definition of $|||\cdot|||$ and using (7.29), we get

$$\begin{aligned} |||v|||^2 &\leq B(v, v) + \mathcal{K}^2 B(v, v) + \|v\|^2_{\frac{1}{2},h} \\ &\leq (1 + \mathcal{K}^2)[B(v, v) + \|v\|^2_{\frac{1}{2},h}]. \end{aligned}$$

Thus combining the above inequality with (7.37) we get

$$B_\gamma(v, v) \geq C^* |||v|||^2,$$

where $C^* = \frac{1}{1+\mathcal{K}^2} \min(\frac{\gamma-\mathcal{K}^2}{\gamma+\mathcal{K}^2}, \frac{\gamma-\mathcal{K}^2}{2})$, which is (7.36). □

We now present the following result:

Theorem 7.4 *Suppose $u_0 \in H^l(\Omega)$, $l \geq 2$ is the solution of (7.8). Let $u_{h,\gamma} \in \mathbb{V}^{k,q}_{\Omega,h}$, with $q \geq 2$, be the solution of (7.31), where $\mathbb{V}^{k,q}_{\Omega,h}$ satisfies (7.28), (7.29). Then*

$$\|u_0 - u_{h,\gamma}\|_{H^1(\Omega)} \leq \frac{C(1+\gamma)}{C^*(X^\nu)} h^\mu \|u_0\|_{H^l(\Omega)}, \quad \mu = \min(k, l-1), \tag{7.38}$$

where $C^*(X^\nu)$ is as in (7.36).

Proof. It is easy to see that

$$B_\gamma(u_0, v) = \int_\Omega f v \, dx, \quad \text{for all } v \in H^1(\Omega),$$

and therefore,

$$B_\gamma(u_0 - u_{h,\gamma}, v) = 0, \quad \text{for all } v \in \mathbb{V}^{k,q}_{\Omega,h}. \tag{7.39}$$

Now for any $g_h \in \mathbb{V}^{k,q}_{\Omega,h}$, using (7.36), (7.39), and (7.35) we have

$$\begin{aligned} |||g_h - u_{h,\gamma}|||^2 &\leq \frac{1}{C^*} \hat{B}_\gamma(g_h - \hat{u}_{h,\gamma}, g_h - u_{h,\gamma}) \\ &\leq \frac{1}{C^*} \hat{B}_\gamma(g_h - u_0, g_h - u_{h,\gamma}) \\ &\leq \frac{(1+\gamma)}{C^*} |||u_0 - g_h|||\, |||g_h - u_{h,\gamma}|||, \end{aligned}$$

and hence

$$|||g_h - u_{h,\gamma}||| \leq \frac{(1+\gamma)}{C^*} |||u_0 - g_h|||.$$

Therefore,

$$\begin{aligned}|||u_0 - u_{h,\gamma}||| &\leq |||u_0 - g_h||| + |||g_h - u_{h,\gamma}||| \\ &\leq C|||u_0 - g_h|||, \quad \text{for all } g \in \mathbb{V}^{k,q}_{\Omega,h}. \end{aligned} \tag{7.40}$$

Now using (7.28) and a trace inequality, we have

$$\begin{aligned}\|u_0 - g_h\|^2_{\frac{1}{2},h} &= \sum_{\underline{x}\in A^h_{\partial\Omega}} h^{-1}_{E^h_{\underline{x}}} \|u_0 - g_h\|^2_{H^0(E^h_{\underline{x}})} \\ &\leq Ch^{-1}\|u_0 - g_h\|^2_{H^0(\partial\Omega)} \\ &\leq Ch^{-1}\left(\frac{1}{h}\|u_0 - g_h\|^2_{H^0(\Omega)} + h\|u_0 - g_h\|^2_{H^1(\Omega)}\right) \\ &= C\left(h^{-2}\|u_0 - g_h\|^2_{H^0(\Omega)} + \|u_0 - g_h\|^2_{H^1(\Omega)}\right), \end{aligned} \tag{7.41}$$

where C depends on κ. Also using a similar argument, we have

$$\|\frac{\partial(u_0 - g_h)}{\partial n}\|^2_{-\frac{1}{2},h} \leq C\left(h^2\|u_0 - g_h\|^2_{H^2(\Omega)} + \|u_0 - g_h\|^2_{H^1(\Omega)}\right). \tag{7.42}$$

where C depends on κ. Thus from (7.41) and (7.42) we get,

$$\begin{aligned}|||u_0 - g_h|||^2 &= \|u_0 - g_h\|^2_{H^1(\Omega)} + \|\frac{\partial(u_0 - g_h)}{\partial n}\|^2_{-\frac{1}{2},h} + \|u_0 - g_h\|^2_{\frac{1}{2},h} \\ &\leq Ch^{-2}\sum_{j=0}^{2} h^{2j}\|u_0 - g_h\|^2_{H^j(\Omega)}, \end{aligned}$$

and hence from the definition of $|||\cdot|||$ and (7.40), we have

$$\begin{aligned}\|u_0 - \hat{u}_{h,\gamma}\|_{H^1(\Omega)} &\leq |||u_0 - \hat{u}_{h,\gamma}||| \\ &\leq Ch^{-1}\sum_{j=0}^{2} h^j\|u_0 - g_h\|_{H^j(\Omega)}, \quad \text{for all } g_h \in \mathbb{V}^{k,q}_{\Omega,h}. \end{aligned} \tag{7.43}$$

Finally, we choose $g \in \mathbb{V}^{k,q}_{\Omega,h}$ such that

$$\|u_0 - g\|_{H^s(\Omega)} \leq Ch^{\mu_1 - s}\|u_0\|_{H^l(\Omega)}, \quad 0 \leq s \leq 2,$$

where $\mu_1 = \min(k+1, l)$. Using this in (7.43) we get the desired result.$\square$

We now discuss the situations where the assumption required to prove Theorem 7.4 are satisfied. The major problem is to estimate $\mathcal{K}(X^h)$ given in (7.29). We would like to have $\mathcal{K}(X^h) \leq C$, uniformly for all $0 < h \leq 1$. If the supports $\eta^h_{\underline{x}}$ of the particle function $\phi^h_{\underline{x}}$ are "reasonable", *e.g.,* circles in $\mathbb{R}^2$ or spheres in $\mathbb{R}^3$ or similar, then it is easy to see that the necessary condition for $\mathcal{K}(X^h) \leq C$ is that, $h_{E^h_{\underline{x}}} \geq \alpha|\eta^h_{\underline{x}}|$ for $\underline{x} \in A^h_{\partial\Omega}$. This can be enforced by properly selecting the set of particles X^h. This aspect can also affect the design of adaptive meshless (Nitsche) method. Since the estimates of these constants difficult to estimate accurately, we may select larger value of γ in (7.31) so that the Theorem 7.4 is valid.

The Nitsche method presented here is superior to both the penalty method and Lagrange multiplier method. Nitsche method, in the framework of meshless methods, was addressed in [15] and implemented in [76] and [46].

4. The Collocation Method

Collocation method, in the framework of meshless methods, was recently proposed in [3, 89, 47]. The method consists of adding constraint equation, at certain points of the boundary $\partial\Omega$, to the stiffness matrix. No analysis was presented to address the convergence of the approximate solution obtained from this method.

5. Combination of Meshless Methods and the Finite Element Method

This method was proposed, *e.g.,* in [51]. The main idea in this approach is to use classical finite elements (which could also be interpreted as particle functions) in a neighborhood of the boundary $\partial\Omega$, and to select other particle functions such that their supports do not intersect $\partial\Omega$.

6. Characteristic Function Method

The method was proposed in connection to Ritz method when the approximating functions were global polynomials (see [63],[50]). If a domain Ω has a smooth boundary $\partial\Omega$, there exists a smooth function Φ such that

$$\begin{aligned} \Phi(x) &> 0, \quad x \in \Omega, \\ \Phi(x) &= 0, \quad x \in \partial\Omega, \\ \text{and } |\nabla\Phi(x)| \geq \alpha &> 0, \quad x \in \partial\Omega. \end{aligned}$$

Let $S_h^\Phi = \{u : u = \Phi v, \ v \in \mathbb{V}_{\Omega,h}^{k,q}\}$. Then it is obvious that $S_h^\Phi \subset H_0^1(\Omega)$. We approximate the solution u_0 of (2.1) and (2.3) by $u_h \in S_h^\Phi$, where u_h is the solution u_S of (2.7) with $S = S_h^\Phi$.

For $u_0 \in H^l(\Omega) \cap H_0^1(\Omega), \ l \geq 2$, we define $w_0 = \frac{u_0}{\Phi}$. Then using Hardy's inequality (Thm. 329 of [49]), one can show that $w_0 \in H^{l-1}(\Omega)$. Using this result, we obtain the following theorem.

Theorem 7.5 *Suppose $u_0 \in H^l(\Omega) \cap H_0^1(\Omega)$, and suppose $l \geq 2$. Then there exists $w_h \in \mathbb{V}_{\Omega,h}^{k,q}$ such that $g_h = \Phi w_h$ satisfies*

$$\|u_0 - g_h\|_{H^1(\Omega)} \leq Ch^\mu \|u_0\|_{H^l(\Omega)}, \quad \mu = \min(k, l-2). \tag{7.44}$$

Proof. Recall that $\mathbb{V}_{\Omega,h}^{k,q}$ is $(k+1,q)$-regular with $q \geq 1$. Then there exists $w_h \in \mathbb{V}_{\Omega,h}^{k,q}$ such that

$$\|w_0 - w_h\|_{H^1(\Omega)} \leq Ch^\mu \|w_0\|_{H^{l-1}(\Omega)} \leq Ch^\mu \|u_0\|_{H^l(\Omega)}, \tag{7.45}$$

where $\mu = \min(k, l-2)$. Now from the definition of w_0, we have

$$u_0 - \Phi w_h = u_0 - \Phi w_0 + \Phi(w_0 - w_h) = \Phi(w_0 - w_h),$$

and hence, using (7.45), we have

$$\|u_0 - \Phi w_h\|_{H^1(\Omega)} \leq C\|w_0 - w_h\|_{H^1(\Omega)} \leq Ch^\mu \|u_0\|_{H^l(\Omega)}, \quad \mu = \min(k, l-2),$$

which is the desired result. $\square$

Remark 7.3 It is clear from (2.8) and (7.44) that for $l \geq 2$,

$$\|u_0 - u_h\|_{H^1(\Omega)} \leq Ch^\mu \|u_0\|_{H^l(\Omega)}, \quad \mu = \min(k, l-2).$$

We further note that this order of convergence, or (7.44) in the last theorem, cannot, in general, be improved.

7. The Generalized Finite Element Method

We note that all the methods described so far primarily use $\mathbb{V}_{\Omega,h}^{k,q}$ as the approximating space. The GFEM, on the other hand, uses different approximating spaces, as we have seen in Section 6. The use of these approximating spaces makes GFEM extremely flexible.

We recall that in GFEM, we start with a partition of unity with respect to a simple mesh that need not conform to the boundary of the domain. This partition unity could be the particle shape functions defined in Section 3. Then "handbook" functions are used as local approximating spaces. The Dirichlet boundary condition can be implemented by choosing the local approximation space $V_{\underline{x}}$, for $\underline{x} \in A_{\partial\Omega}^h$, such that the functions in $V_{\underline{x}}$ satisfy the Dirichlet boundary conditions.

We presented a few approaches on how to use meshless approximation to approximate solutions of PDE's. To impose Dirichlet boundary conditions on meshless approximation is a challenge, and we looked into some methods that can overcome this difficulty. While discussing these methods, we assumed that the boundary of the domain is smooth for simplicity. But the results presented here can be generalized to include non-smooth boundary, especially, piecewise smooth boundary.

Some of methods were implemented and reported in the literature, but they lacked rigorous theoretical analysis. All the methods reported here have certain advantages as well as disadvantages. If the particle space $\mathbb{V}_{\Omega,h}^{k,q}$ is used as an approximating space, in our opinion, the Nitsche method is very promising because it is robust relative to other methods and is easy to implement. But we note that $\mathbb{V}^{k,q}_{\Omega,h}$ is difficult to construct for higher values of k, and the use of $\mathbb{V}^{k,q}_{\Omega,h}$ with lower values of k reduces the accuracy of the method. On the other hand, GFEM uses a partition of unity (*e.g.,* $\mathbb{V}^{k,q}_{\Omega,h}$ with $k = 0$), which is easy to construct, and higher accuracy can be attained by using suitable local approximation spaces.

8 Implementational Aspects of the Meshless Method

In this section, we will briefly discuss the implementational aspects of meshless methods and the GFEM. Similar to the finite element method, the implementation of meshless methods and the GFEM has four major parts:

1. Construction of particle shape functions.
2. Construction of the stiffness matrix.
3. Solution of the linear system of equations.
4. A-posteriori error estimation, adaptivity, and computation of data of interest.

We now discuss these items in turn.

1. Construction of particle shape functions

In the classical finite element method, one starts with a mesh that is related to the domain, and then shape functions are defined with respect to the chosen mesh. In a meshless method, one starts with particles $\{\underline{x}\}$. Corresponding to each particle $\underline{x}$, a particle shape function with compact support $\eta_{\underline{x}}$ is constructed, such that $\mathring{\eta}_{\underline{x}}$'s form an open cover of the domain Ω. The construction of shape functions that are reproducing of order $k = 0$ or 1 is not difficult. For $k = 0$, one may use Shepard's approach ([77]) as described in Section 4.1. For $k = 1$ and for an appropriate particle distribution, one may first construct a mesh using tetrahedrons such that the particles are the nodes of the mesh (*i.e.,* the vertices of the tetrahedrons). This procedure is not difficult as there are efficient codes available for constructing such a mesh. The shape function corresponding to the particle $\underline{x}$ can be taken to be the standard hat functions, whose support is the union of all the simplices with $\underline{x}$ as one of its vertices. We note however that, for $k = 1$, smoother shape functions can also be constructed (see [48]). For $k = 0,\ 1$, one has to check that $\mathrm{card}(S_{\underline{x}}) \leq \kappa$, κ is independent of $\underline{x}$, where $S_{\underline{x}} = \{\underline{y} : \eta_{\underline{y}} \cap \eta_{\underline{x}} \neq \emptyset\}$. For the Nitsche Method, described in Section 7.2, one also has to check that $\mathcal{K}$, defined in (7.29), is bounded. The construction of particle shape functions for $k \geq 2$ is more expensive than for $k = 0, 1$, and it may be more difficult to check assumptions A1–A7 and (3.64), which ensure convergence.

In contrast, the GFEM uses only a partition of unity, and thus particle shape functions with $k = 0, 1$, described in the last paragraph, can be used for this purpose. Also, a simple regular distribution of particles could be used to construct the partition of unity. The space of local shape functions, $V_{\underline{x}}$, could be created analytically or through "handbook" solutions. Dirichlet boundary condition is also treated by an appropriate selection of $V_{\underline{x}}$, and hence one does not have to use special methods, *e.g.,* the penalty method, Nitsche method, etc., which simplifies the implementation.

2. Construction of the stiffness matrix

The construction of the stiffness matrix for a meshless method is laborious and delicate. In fact, this is where one pays the price for avoiding mesh generation. The elements of the resulting stiffness matrix are integrals, which have to be numerically evaluated over various regions. These regions are not simple tetrahedrons as in the finite element method, where they naturally come from a mesh. These regions, for a meshless method, are of the form $\eta_{\underline{x}} \cap \eta_{\underline{y}} \cap \Omega$, $\underline{x}, \underline{y} \in X^\nu$, and can be extremely complicated. Also the integrals have to be evaluated accurately as it is known that inaccurate numerical integration leads to very poor results (see, for example, [28]). A special numerical integration scheme is given in [30], where $\eta_{\underline{x}}$'s are spheres and the region of integration are the intersection of two spheres. The problem of effective integration has also been addressed in [33, 43, 76, 82, 83]. The numerical integration poses additional problems in GFEM when singular functions are included in the local approximating space $V_{\underline{x}}$. Standard integration schemes in this situation yield poor accuracy. This problem in GFEM was handled in [82] by using adaptive numerical integration. Because of this sensitivity to numerical integration, the use of adaptive integration is preferred, in general, in GFEM.

Thus we see that an accurate and effective numerical integration scheme to approximate the elements of the stiffness matrix is essential for the success of meshless methods. We will further remark on this issue in the next item of this section. We note that numerical integration and construction of stiffness matrix in these methods are parallelizable.

3. Solution of the linear system

The exact stiffness matrix (without numerical integration) obtained from a meshless method could be positive definite with a large condition number. This is caused by using shape functions with large overlap between their supports, which makes the shape functions "almost" linearly dependent. Moreover, the exact stiffness matrix obtained from GFEM could be positive semi-definite, as shown in [82]. But the underlying linear system obtained from the GFEM is always consistent, *i.e.,* the linear system has non unique solutions. The lack of unique solvability of the linear system does not imply that the GFEM produces non-unique solutions. In fact, if the vector $\{c_{\underline{x},j}\}_{1\le j\le n_{\underline{x}}}$, with $\dim V_{\underline{x}} = n_{\underline{x}}$, is a solution of this linear system, then the solution,

$$u_h = \sum_{\underline{x}} \sum_{j=1}^{n_{\underline{x}}} \phi_{\underline{x}}\, c_{\underline{x},j}\, \psi_{\underline{x}}^j,$$

obtained from the GFEM, where $\psi_{\underline{x}}^j$ is a basis of $V_{\underline{x}}$, is unique.

We already have mentioned the importance of numerical integration in evaluating the elements of the stiffness matrix obtained from a meshless method. We further note that the elements of the load vector is also evaluated by numerical integration. To obtain a consistent linear system (after the use of numerical integration), the numerical integration scheme applied to compute an element of the load vector should be same as the scheme used to compute the corresponding row of the stiffness matrix.

To find the solution of the linear system obtained from a meshless method (or from the GFEM), one can use a special direct solver based on elimination or an iterative solver. In [82], direct solvers, *e.g.,* subroutines MA27 and MA47 of *Harwell Subroutine Library*, was used to solve the sparse positive semi-definite linear system obtained from GFEM. The use of these solvers was successful even when the nullity of the stiffness matrix was large. It was also shown in [82] that round-off errors did not play a significant role in solving the linear system, *i.e.,* the round-off error was almost same as when the standard finite element linear system is solved by the elimination method.

An iterative algorithm for solving such linear systems was given in [82]. The idea of this algorithm is to perturb the stiffness matrix by a unit matrix multiplied by a small parameter. The perturbed matrix, say P, is positive definite and any solver could be used to solve $Px = b$. Using this fact and a few iterations of a simple iterative technique, a solution of the original linear system could be obtained. We refer to [82] for a complete description effectiveness of this iterative algorithm.

We have noted before that the linear system obtained from the meshless method is consistent even if the stiffness matrix is positive semi-definite. In this situation, a solver based on conjugate gradient method can also be used. The convergence in this situation is similar to the convergence of conjugate gradient method when applied to solve the linear system obtained from the standard finite element method. The multigrid method is not directly applicable to the linear system when the stiffness matrix is positive semi-definite, since the eigenfunctions corresponding to the zero eigenvalue of the stiffness matrix is global and oscillatory. The same is also true when the the stiffness matrix is "almost" singular. However, a special version of multigrid method was proposed in [76] and [44], when the underlying partition of unity was reproducing of order $k = 0$.

4. A-posteriori error estimation, adaptivity, and computation of data of interest

The rigorous theory of a-posteriori error estimation was developed in [17] and other estimates, based on various averaging, were also used. These estimators can be used as error indicators for adaptivity purposes. For adaptive approaches in meshless method, we refer the reader to [76] and [23].

We finally mention that programming the meshless method is an important issue and it requires specific concepts. For this aspect of the meshless method, we refer to [32, 45, 82] and [83].

9 Examples

Meshless methods have been applied to linear and non linear elliptic problems, as well as to problems related to other differential equations; we refer to [56]. However, it is essential to characterize the types of problems where this method is, or could be, superior to standard methods ([21]).

In this paper, we address only the application of the meshless method on a class of linear, elliptic problems. As stated in the introduction, one of the main advantages of meshless methods is that it avoids mesh generation. This is essential when the domain in complex. Another advantage of this method is that it allows the use of various "special" local shape functions to improve the accuracy.

The Generalized Finite Element Method (GFEM) was elaborately discussed in [83] and it was shown that the method is effective. Three types of meshes with successive refinements were used in that paper and we present one of these meshes in Figure 9.1. This is a simple finite element mesh and it *does not* reflect the geometry of the underlying domain. Then, using the linear finite elements as partition of unity and special functions for local approximation, an improvement in the rate of convergence was achieved. Detailed numerical data, with comments on various aspects of the method, *e.g.,* numerical integration, *etc.* were presented in [83]. We note however, that though the domain considered in this example (*i.e.,* the domain in Figure 9.1) was simple, and classical finite element method with mesh refinement or an adaptive procedure could have been used, the analysis and data presented in [83] clearly shows the scope and potential of GFEM.

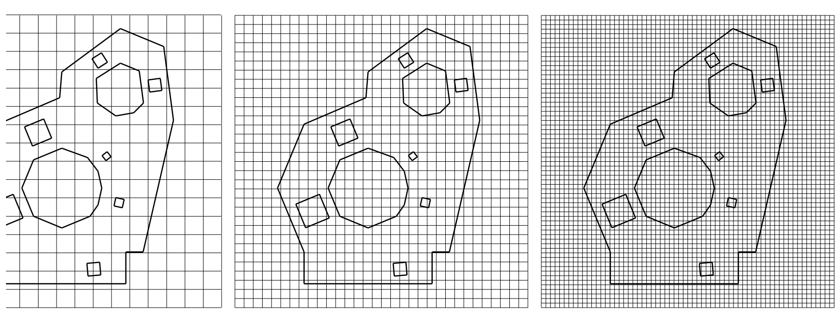

Figure 9.1: A mesh used in [83] for the constructing of partition of unity in the context of GFEM to solve a problem posed on the domain.

As mentioned above, the power of GFEM lies in handling problems where the underlying domain has complex geometry. Three types complex domains, shown in Figure 9.2, were analyzed by the GFEM in [83]. Another complex domain with fibers, analyzed in the same paper, is shown in Figure 9.3 where the fiber distribution was taken from [10]. To construct finite element meshes for these domains is very complex and nearly impossible. “Handbook” problems that characterize the local behavior of the approximated solution (*e.g.,* in the neighborhood of a crack, fibers, etc.) were used to construct special shape functions for these problems.

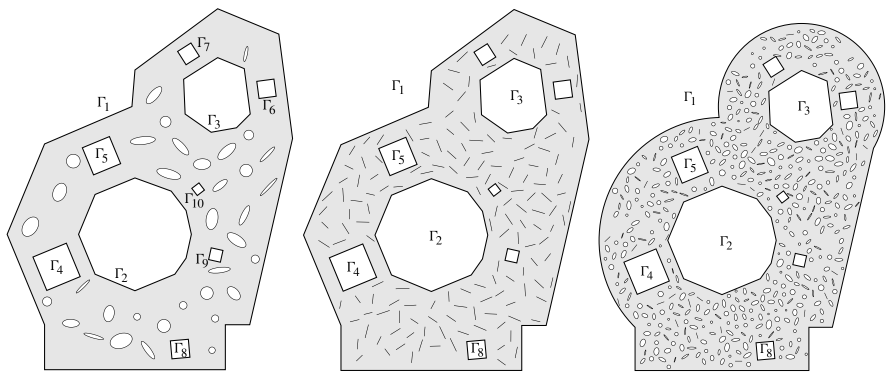

Figure 9.2: The three types of domains analyzed by GFEM in [83].

The GFEM has an advantage in dealing problems with singularities (in the neighborhood of geometric edges) in 3D. When the basic finite element tetrahedral mesh is used in such problems, it is well known that classical edge refinement is cumbersome. This problem was handled using GFEM in [36], where a refinement by special functions, at positions indicated by an error indicator, was performed. GFEM was also used to handle difficulties stemming from orthotropic problems in [35].

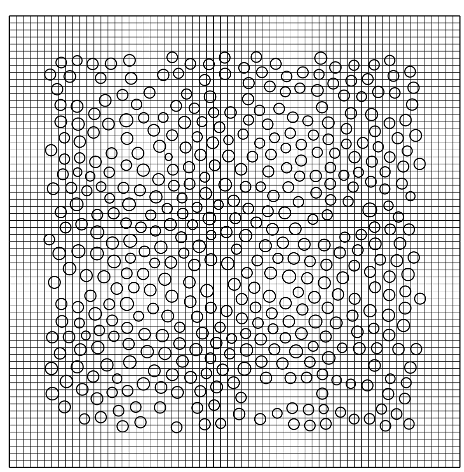

Figure 9.3: The problem of fiber composite type analyzed in [83].

There are other types of problems where the GFEM is quite effective. They include multisite problems, where many crack configuration are present, and crack propagation problems, where the geometry of the domain changes. The GFEM could be used in such problems by considering local approximating spaces consisting of functions that are discontinuous over the cracks and including a singular function with respect to the tip of the crack into the local approximating space. Then the propagation of the crack is computed (using stress intensity factors); the old singular function in the approximating space is replaced by a new singular function, new discontinuous functions are added in the same space, and the process of computing the propagation of crack and changing the local approximation spaces is repeated. In this process, the matrix of the underlying linear system at a particular step can be obtained by augmenting the matrix corresponding to the previous step with new rows and columns, and it is possible to solve the new linear system using Schur complement, which uses the previously computed data. This general idea was used in [56, 64] without using the previously computed data.

The GFEM is an important tool in approximating solutions of elliptic problems with rough coefficients as well as homogenization problems. We mention that the usual finite element method may give extremely poor result when applied to elliptic problems with rough coefficients, as shown in [18]. It was shown in [16], using a detailed analysis, that GFEM leads to the same rate of convergence for problems with rough coefficients as when the coefficients are smooth.

We emphasize that in this paper, we considered only a small (but important) family of problems. We showed that the use of meshless methods, particularly the use of GFEM, on these problems is advantageous in comparison to the standard finite element method. Of course, there are other types of problems, especially non-linear problems, which we have not addressed in this paper.

Note: The authors will like to thank Elsevier Science for allowing the use of Figures 9.1, 9.2, and 9.3 in this paper. They were published in the journal *Computer Methods in Applied Mechanics and Engineering.*

10 Some Comments and Future Challenges

In the previous sections we addressed linear elliptic equations. The approximation theory we developed is obviously usable in practically all variationally formulated problem, provided the inf-sup condition (the BB condition) is satisfied. Meshless methods and the GFEM can be directly used for higher order equations because there are no difficulties in constructing shape functions of any regularity. Adaptive procedures, shock capturing, *etc.*, can be combined with adaptive construction of shape functions. A deep theoretical understanding of these issues is still in the future.

For non-coercive problems, the question of the inf-sup condition is more delicate and plays an important role. Some non-coercive problems have been successfully treated. We mention, for example, equations whose solutions are oscillatory, as with the Helmholtz equation. Special shape functions were used to capture the oscillatory character of the solution; see, *e.g.* [52], [62]. Although there are some problems with round-off error, it can be expected that these difficulties can be overcome.

Meshless Methods can be applied to nonlinear as well as linear problems. The Handbook approach for constructing shape functions is important for both types of problems.

The Finite Element Method is presently used for solving a wide variety of problems. Meshless methods, especially the GFEM, offer many new opportunities. Since the FEM is a special case of the GFEM, the study of the GFEM provides the potential to develop enhanced and improved methods. Under the umbrella of GFEM, there is actually a family of particular methods—depending on the approximating functions employed. The goal in creating new particular GFEM is to obtain methods that are especially effective on certain classes of problems. To address this issue meaningfully and effectively, further mathematical study of meshless methods is essential.

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